A Topological proof of Abel-Ruffini Theorem - Henryk Zoladek

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The Abel-Ruffini theorem which states that any general algebraic equation of degree ≥ 5 cannot be solved in radicals is one of the most celebrated results in algebra. This means that there does not exist any formula to compute the roots of a general polynomial like there exists for quadratic and cubic equations.

The revolutionary proofs of Abel and Galois following Ruffini paved way for group theory and Galois theory. and thus the traditional route to proving this theorem is through Galois theory but the author in this paper provides a refreshingly new proof through complex analysis using the topological properties of Riemann surfaces. This proof originally by V.I. Arnold forms the starting point of a field called topological Galois theory.

An algebraic function y=f(x) can be seen as solution sets of the algebraic equation

$$F(x,y) = g_n(x)y^n + g_{n-1}y^{n-1} + \cdots + g_0(x) = 0$$

where g_j are polynomials. For example, the algebraic function $y = \sqrt[2]{x}$ can be seen as the root of the equation $y^2 - x = 0$.

Since it is a polynomial in y it can be written as $\Pi_{i=1}^{n}(y - f_i(x))$ The Implicit Function Theorem guarantees that these n roots are distinct in the neighbourhood of a complex number a if $F'_y(a, f_i(a)) \neq 0 \quad \forall i \in \{1, 2, ..n\}$.These roots are called the analytic elements of f.

Let us look at the previous example of $y^2 - x = 0$.
Now at any point other than $x = 0$ the condition is satisfied and the equation splits into 2 elements i.e $y = \sqrt[2]{x}$ and $y = -\sqrt[2]{x}$. Now consider the unit circle around x=0. If we evaluate $\sqrt[2]{x}$ around the loop going from $\theta = 0$ to π we obtain $\sqrt[2]{x} = e^{\frac{i\theta}{2}} = -e^{\frac{i(\theta+2\pi)}{2}}$.

Thus, we can see that one analytic element transforms into the other after one rotation around the loop. Had we taken a loop which did not enclose $x = 0$ then the value wouldn't have changed.

In general any loop defines a permutation on the set $M_a = \{f_i(a), ..., f_n(a)\}$. The group defined by the set of permutations arising due to the various loops is the monodromy group of f.

The monodromy group of the function f is clearly a subgroup of the group S(n)(group of permutations of set of n elements). The paper shows that the monodromy group of an algebraic equation solvable by radicals is solvable by showing that if the groups Mon(f) and Mon(g) are solvable, then the groups Mon($f \pm g$), Mon(fg), Mon(f /g) and Mon($k^{1/2}f$) are also solvable.

Assume that there exists a formula to solve a n-degree polynomial that is a radical function in the coefficients then the polynomial $3y^5 - 2y^3 + 60y^3 - x = 0$ is solvable by radicals but its monodromy group is S_5 which is not solvable and this is a contradiction.