Sheaf Cohomology

MTH661 - Topos Theory

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1 What is this?

This is a hodgepodge of definitions and theorems picked up from various sources to give the reader a brief introduction to the area of sheaf cohomology. Therefore, none of the material presented here is an original work of mine and no such assumption should be made even if a citation isn't provided. As this is a project report for a topos theory course, background in category theory is assumed but nothing more i.e all definitions related to cohomology are provided. This exposition also omits most proofs and instead points to the sources where they are available not because the proofs are hard or require extra machinery (infact most are just routine diagram chasing arguments often seen in categorical proofs) but because including them would make it significantly voluminous and there is no point in reproducing them. I hope the reader will find this text helpful. This has been written in haste and the author is solely responsible for (m?)any errors that might be present.

2 Abelian Sheaves

In the course we have only considered sheaves over the category of sets but now we shift focus to those over abelian groups (and then to modules). The important observation that is used in most of the proofs is that the stalks uniquely determine a section i.e for any sheaf $\mathcal{F}$ $s, t \in \mathcal{F}(U), s = t \iff s_x = t_x \; \forall x \in U$

This lemma can be used to extend our set-based construction to incorporate algebraic operations. For example, the ΓL functor as we defined it was a set, i.e $\Gamma LF(U) := \{\sigma | p \circ \sigma = id\}$

Definition 2.1. A *ringed space* $(X, \mathcal{O}_X)$ is a topological space X together with a sheaf of rings $\mathcal{O}_X$ on X. The sheaf $\mathcal{O}_X$ is called the *structure sheaf* of X.

Definition 2.2. An $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ such that $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$-module and the restriction maps are module homomorphisms. The category of all such modules is referred to as $Mod_{\mathcal{O}_X}(X)$ or just $Mod(X)$

An easy observation is that any sheaf over abelian groups is an $\mathcal{O}_X$ module where $\mathcal{O}_X(U) = \mathbb{Z}$ is the constant sheaf. Thus,

$$Ab(X) \xhookrightarrow{\iota} Mod(X)$$

As expected, the category $Mod(X)$ is abelian. That is it is additive is easy to see by defining addition stalk-wise. The kernel is also easy to construct but construction of the image requires sheafification. For a proof see [9, Tag 01AF]

3 Enough (of) Injective Objects

Having enough injectives is an essential property to construct cohomology groups and we know that a topos has enough injectives. In this section, we will review the definitions and see that the category $Mod(X)$ has enough injectives too.

Definition 3.1. An object $I \in Ob\,\mathscr{C}$ is injective if for any monomorphism $m : A \to B$ and $f : A \to I \;\; \exists g : B \to I$ such that $g \circ m = f$

Definition 3.2. An category is said to have *enough injectives* if for every objext $A \;\; \exists m : A \to I$ where m is a monomorphism and I is an injective object.

Lemma 3.1. *$\prod_i A_i$ is injective iff A_i is injective $\forall i$*

Proof. $Hom(\_, I)$ is a left exact functor as it is right adjoint to the $I \otimes \_$ functor (recall Hom- tensor adjunction). The right exactness holds as the requirement is that the map induced by the mono $m : A \to B$ from $Hom(B, I) \to Hom(A, I)$ be surjective which follows easily from the definition of injective objects. Now, $Hom(\_, \prod_i A_i) \cong \prod_i Hom(\_, A_i)$ which is exact iff each of the functors are which happens iff A_i is injective $\forall i$. ☐

Let us see the ramifications of this in an Abelian category

Theorem 3.2. *If R is a commutative ring with unity, the category $R-Mod$ of R- Modules has enough injectives.*

Proof. We will just sketch an outline of the proof. Refer to [10] (or [8] for an explicit one).

1. Characterization of an injective group - G is injective in (Ab) iff G is divisible (i.e $\forall g \in G, \forall n \in \mathbb{Z}^* \ \exists h \in G, g = nh$)
2. Prove that if G is injective then for any ring R, the R-module $Hom_R(R, G)$ which is the R-module of abelian group homomorphisms from R to G where for each $f \in Hom_R(R, G)$ the multiplication is defined by $(r.f)(s) := f(rs) \ \ r, s \in R,$
3. Take the injective group G to be $\mathbb{Q}/\mathbb{Z}$ which is also the cogenerator in the category of Abelian groups.
4. Show that there is in injection of every R-module F into $\prod_{\phi \in Hom_R(F,E)} E$ where E is any injective module (more specifically it is taken to be $Hom_R(R, \mathbb{Q}/\mathbb{Z})$)

□

Theorem 3.3. *$Mod(X)$ has enough injectives.*

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$ module. Its' stalks $\mathcal{F}_x$ are $\mathcal{O}_{X,x}$ modules and by the previous theorem can be embedded into an injective $\mathcal{O}_{X,x}$ module I_x. Construct $I(U) = \prod_{x \in U} I_x$. This is injective as for any given mono $A \to B$, just look at the stalks, construct the maps $h_x : B_x \to I_x$ and these then give a map $h : B \to I$ by the universal property □

4 Homological Algebra

In this section we will define the necessary concepts that we need from homological algebra.

Definition 4.1. A *(cochain) complex* $A^\bullet$ in an abelian category $\mathscr{A}$ is a collection of objects A_i of $\mathscr{A}$, $i \in \mathbb{N}$, together with morphisms $d_i : A_i \to A_{i+1}$ such that $d_{i+1} \circ d_i = 0 \ \ \forall i$. The maps d_i are called the *differential* maps of the complex $A^\bullet$.

Definition 4.2. The n^{th} cohomology group written $H^n(A^\bullet) :=$ Ker $d_n/$Im d_{n-1}. A complex $A^\bullet$ is *exact* if $H^n(A^\bullet) = 0 \ \ \forall n \geq 1$

Definition 4.3. A pair of morphisms $f, g : A^\bullet \to B^\bullet$ is *chain-homotopic* if there are a set of morphisms $k_i : A_{i+1} \to B_i$ such that $f_i - g_i = d_{B,i-1}k_{i-1} + k_i d_{A,i} \ \ \forall i \in \mathbb{N}$

Lemma 4.1. *A morphism $f : A^\bullet \to B^\bullet$ induces a map between the cohomology groups. Moreover, these maps are equal if the morphisms are chain homotopic*

Definition 4.4. Given $A \in \mathscr{A}$, a complex $A^\bullet$ is said to be a *resolution* of A if $H^0(A^\bullet) = A$ and it is exact. Moreover, if each of the A_i is injective it is called an *injective resolution*.

Let us use the fact that $Mod(X)$ has enough injectives to create an injective resoltion of an arbitrary object A. First we create a resolution by taking injections, and then the cokernel and then the injection and so on. We then only retain the injectives in the chain by taking compositions.

$$0 \xrightarrow{!_A} A \xrightarrow{i_0} I^0_A \xrightarrow{c_0} Coker(i_0) \xrightarrow{i_1} I^1_{Coker(i_0)} \cdots$$

$$0 \xrightarrow{!_A} A \xrightarrow{i_0} I^0_A \xrightarrow{i_1 \circ c_0} I^1_{Coker(i_0)} \cdots$$

$$0 \longrightarrow A \longrightarrow I^\bullet$$

Wherin the last representation is just a compact notation that can be read as $I^\bullet$ is an injective resolution of A. Note that this is not necessarily unique as an object can be embedded into many injective objects. Now, we see an important result which allows us to define homology groups of an object.

Theorem 4.2. *Let $I^\bullet$ (resp. $J^\bullet$) be an injective resolution of an object A (B). Then any morphism $f : A \to B$ induces a morphism of complexes $f^\bullet : I^\bullet \to J^\bullet$, which is unique up to homotopy.*

Corollary 4.2.1. *The folllowing hold for $R^iF(A) = H^i(FI^\bullet)$ where F is any left-exact functor.*

1. *$R^iF(A)$ is well defined*
2. *$R^0F(A)$ is naturally isomorphic to F*
3. *If A is injective, $R^iF(A) = 0 \;\; \forall i > 0$*

Proof. 1. Taking $A = B$ in the theorem and taking the map $f = id_A$, we get that any other map is other map is homotopy equivalent to this. Clearly, the identity on A induces an identity map on the homology groups $H^i(FI^\bullet), H^i(FJ^\bullet)$ and thus, they are isomorphic. This makes sense as from 4.1 homotopy equivalent morphisms induce same maps.

2. Follows directly from the definition of H^0 and the fact that F is left exact and thus preserves limits which gives $H^0(F) = Ker(Fi_0) = FKer(i_0) = FA$.

3. The resolution $I^\bullet$ is just A and $I_i = 0 \; \forall i \; > 0$. $H^0 = FA$, rest are 0.

□

5 ∂ and Derived functors

Definition 5.1. A collection of covariant functors $\{F^i \, | i \geq 0\}$ such that every short exact sequence is extended to a long exact one in a natural way i.e given

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & F^0A & \longrightarrow & F^0B & \longrightarrow & F^0C & \xrightarrow{\partial_0} & F^1A & \longrightarrow & F^1B & \longrightarrow & F^1C & \xrightarrow{\partial_1} & F^2A\cdots \\
 & & \downarrow{\scriptstyle F^0f} & & & & \downarrow{\scriptstyle F^0g} & & \downarrow{\scriptstyle F^1f} & & & & \downarrow{\scriptstyle F^1g} & & \\
0 & \longrightarrow & F^0A & \longrightarrow & F^0B & \longrightarrow & F^0C & \xrightarrow{\partial_0} & F^1A & \longrightarrow & F^1B & \longrightarrow & F^1C & \xrightarrow{\partial_1} & F^2A\cdots
\end{array}$$

Definition 5.2. A set of definitions related to $\partial-$ functors

1. Exact $\partial-$ functor - : Given any sequence, the extended one is exact.

2. $\partial-$ functor over G: A $\partial-$ functor $\{F^i\}$ such that $G \cong F^0$

3. Universal $\partial-$ functor : $\{F^i\}$ is universal if for any $\{F^i\}$ a morphism $F^0 \to G^0$ can be uniquely extended to a morphism of $\partial-$functors

4. Effaceable $\partial-$functor : $\{F^i\}$ is effaceable if for any injective object $G^i(I) = 0 \; \forall i > 0$

Now we state one of the most crucial theorem of this section. The proof involves a lot of messy diagram chasing and can be seen in [10], outline of a (partial) homological proof can be found in [4] which uses snake lemma and horseshoe lemma.

Theorem 5.1. *If $F : \mathscr{C} \to \mathscr{D}$ is a left exact functor between abelian categories where $\mathscr{C}$ has enough injectives, then the $R^\bullet = \{R^iF\}$ is a universal exact $\partial-$functor over F. Moreover, for any exact effaceable $\partial-$functor $\{G^i\}$ over F, the morphism $\phi : R^\bullet \to G^\bullet$ is an isomorphism of $\partial-$ functors over F.*

6 Sheaf Cohomology is just $R^i\Gamma(X, \_)$

Using 5.1, we can obtain universal exact $\partial-$functor over the global sections functor $\Gamma(X, \_) : Mod(X) \to \mathcal{O}_X(X) - Mod$ (defined by $\Gamma(U, F) := F(U)$) as we have already seen in class that it is left-exact. This gives us $H^i(X, F) := R^i\Gamma(F)$ which is known as the i^{th} derived functor cohomology group.

Similarly, given a morphism of ringed spaces $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ we have a left exact functor $\phi_* : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ which induces a functor between the $Mod(X) \to Mod(Y)$ which is also left exact and thus R^i can be used to obtain a $\partial-$ functor. Surprisingly, however, these two are more related than they might appear.

Theorem 6.1. *If $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, with underlying continous map $\psi : X \to Y$. Let $A \in Ob\ Mod(X)$ and define an $\mathcal{O}_Y$ module $\hat{H}^i_{\psi,V}$ as sheafification of the presheaf that maps $V \to H^i(\psi^{-1}(V), A)$. Then, $R^i\phi_*A \cong H'^i_{\psi,V}$*

But computing the cohomology groups using injective resolutions is very difficult and is rarely done in practice. An easier alternative, is to use what are called flabby sheaves which are over abelian groups. These sheaves are a lesser restricted version of injective sheaves (i.e. all injective sheaves are flabby) and the interesting result is that moving to the abelian group by forgetting the $\mathcal{O}_X$ module doesn't change the cohomology groups. Let's define these sheaves now (from [9, Tag 09SV]).

Definition 6.1. Let X be a topological space. We say a presheaf of sets $\mathcal{F}$ is *flasque* or *flabby* if for every $U \subset V$ open in X the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective.

In fact, a very nice categorical definition just like we had for injective sheaves by defining E-injective sheaves that is they satisfy the lifting condition for a particular set of monomorphisms, E. If E is taken be entire set of monos the definition is that of an injective object. But using the Yoneda lemma one can prove that flasque sheaves can indeed be seen as E-injectives, where E consists of the inclusion maps $\mathcal{O}_X\restriction_U \to \mathcal{O}_X\restriction_V$, for all pairs of opens $U \subset V$. The proof is neatly given in [6] and is reproduced below.

Proof. Let $\mathcal{O}_X\restriction_U$ be the sheaf of modules with $\mathcal{O}_X\restriction_U(V) = \mathcal{O}_X(W)$ if $W \subseteq U$ and 0 otherwise. This is the free $\mathcal{O}_X$-module on the Yoneda sheaf of sets, defined by $y(U)(W) = 1$ if $W \subseteq U$ and $\emptyset$ otherwise. By the Yoneda lemma, $Mod(X)[y(U), F] \cong FU$. For F a sheaf of modules, maps $y(U) \to |F|$ (where $|F|$ is the underlying sheaf of sets of F) correspond to module maps $\mathcal{O}_X\restriction_U \to F$; so elements of $F(U)$ correspond to such maps, and restriction $F(V) \to F(U)$ corresponds to composition with the inclusion map $\mathcal{O}_X\restriction_U \to \mathcal{O}_X\restriction_V$. So this restriction map is surjective exactly if F is injective w.r.t. that inclusion map; and F is flasque exactly if it's injective w.r.t. the set of all such inclusions. □

Theorem 6.2. *If $0 \xrightarrow{d_1} F' \xrightarrow{d_2} F \xrightarrow{d_3} F'' \longrightarrow 0$ is an exact sequence of sheaves and F' is flasque, the following hold,*

1. *For each open U of X, the sequence*
$0 \xrightarrow{d_{1U}} F'(U) \xrightarrow{d_{2U}} F(U) \xrightarrow{d_{3U}} F''(U) \longrightarrow 0$ *is exact.*

2. *If F is flasque, then so is F''*

Since, most of the books leave it as an exercise (including Hartshorne, Tennison,) , we will present the complete proof here. This is referenced from [7]

Proof. Since F is already left-exact proving that $F(U) \to F"(U)$ is surjective suffices. So, let's take a section s of $F"(U)$ and try to construct its preimage. As d_3 is epi we get that, for any open cover $\{V_i\}$ of U $\exists t_i \in F(V_i)$ such that $d_{3U}(t_i) = s\restriction_{V_i}$.

Now the goal is to somehow glue these t_i's together. As, restrictions must agree for a sheaf, $d_{3V_i \cap V_j}(t_i - t_j) = s\restriction_{V_i}\restriction_{V_i \cap V_j} - s\restriction_{V_j}\restriction_{V_i \cap V_j} = 0$. Therefore, $(t_i - t_j)\restriction_{V_i \cap V_j} \in Ker(d_3)$ and by exactness it lies in $Im(d_2)$ i.e. $\exists\ a_ij \in F'(V_i \cap V_2)$ such that $d_2(a_{ij}) = (t_i - t_j)\restriction_{V_i \cap V_j}$.

Since, F' is flabby, we have that $F'(V_i) \to F'(V_i \cap V_j)$ is surjective and we this get $b_i \in V_i$ as the preimage of a_{ij}. Define $t'(i) = t_i, t'_j = t_j + d_2(b_j)$. Since, $d_3 \circ d_2 = 0$, these t' still map to the same s but by naturality of the diagram,

$$\begin{array}{ccc} F'(V_j)(b_j) & \xrightarrow{\quad d_{2V_j} \quad} & F(V_2) \\ \downarrow & & \downarrow \\ F'(V_i \cap V_j)(a_{ij}) & \xrightarrow{d_{2V_i \cap V_j}} & F(V_1 \cap V_2)((t_i - t_j)\restriction_{V_i \cap V_j}) \end{array}$$

We get that $((t'_i - t'_j)\restriction_{V_i \cap V_j}) = 0$. Therefore, by the sheaf property we can glue up these t_i into an element t which is the preimage. For a non finite cover, Zorn's lemma has to be used and for this we ask the reader to consult [7] □

7 Interesting applications

This section will be very vague as the examples we try to present require more jargon and machinery to be made precise but we nevertheless felt it would be informative. Let us first motivate the idea of a quasicoherent (and coherent) sheaf which is a very central concept in algebraic geometry.

7.1 (Quasi)coherent Sheaves

Let S be a graded ring (say, $k[x_0, \cdots, x_n]$ where k is a field) and let M be an S-module. We can create a sheaf on a scheme (a projective scheme Proj S is considered) out of M by constructing the stalks as the localizations at a point p where p is a prime ideal (recall that points in a scheme are prime ideals) i.e. $M_{(p)}$. Such an $\tilde{M}$ is quasicoherent and if S is noetherian and M is finitely generated, it is coherent. See Propostion 5.11 in [5] , There are quite a few ways of defining a quasicoherent sheaf (see [2]) and one of them is that it is locally equal to the cokernel of a morphism between freely generated $\mathcal{O}_X(U)$ modules i.e.

$\mathcal{O}_X^I(U) \xrightarrow{f} \mathcal{O}_X^J(U) \longrightarrow Q(U) \cong Cokernel(f)$ It is coherent if the two modules are finitely generated. See [2] for more details It is to be noted however that as one has to be careful while defining coherent sheaves as the requirement in general (non Noetherian cases) is more than just finite presentability, read section 13.6.6, Page 388 in [11] for a discussion on this.

7.2 Vanishing Theorems

These theorem give us an “application" by saying that we can find out the dimension of a variety by computing the cohomology groups.

Theorem 7.1 (Grothendieck). *If X is a noetherian topological space of dimension n, then* $H^i(X, F) = 0 \ \forall\, i > n$ *and any sheaf of abelian groups F.*

Theorem 7.2 (Serre). *Let X be a noetherian scheme. Then X is affine if and only if for every quasi-coherent sheaf F on X, we have* $H^i(X, F) = 0 \ \forall\, i > 0$.

7.3 Generalizing Euler Characteristic - Algebraic geometrist's way

The classical Euler characteristic $\chi = V - E + F$, has many generalisations ([1] contains a list). For any coherent sheaf $\mathcal{F}$ on a proper scheme X, one defines its Euler characteristic to be :

$$\chi(\mathcal{F}) = \sum_i (-1)^i dim H^i(X, \mathcal{F})$$

In this case, the dimensions are all finite by Grothendieck's finiteness theorem. This is an instance of the Euler characteristic of a chain complex, where the chain complex is a finite resolution of $\mathcal{F}$ by acyclic sheaves.

7.4 Math $\cap$ Complexity theory - Sheaf Cohomology is $\#P$- hard

This is a very interesting work [3] which reduces the problem of counting the number of satisfiable assignments of a 3- CNF boolean formula ϕ which is the canonical $\#P$ complete problem, #SAT, to computing the dimension of a cohomology group (Note that Cech cohomolgy is used). This is done by constructing a coherent sheaf on a projective space as a graded matrix of polynomials defined by the clauses in ϕ for example $x_1 \vee$

$\bar{x_2} \vee \bar{x_3}$ define $(1 - x_1)x_2x_3$ and showing that the degree of the variety define by this set of polynomials give the number of solutions as well as half the dimension of the cohomology group.

$$\#\text{satisfying assingments} = \frac{\text{dim } H^1\left(\mathbb{P}_k^{n+1}, I\right) + 1}{2}$$

where I is the sheaf of of regular functions on $\mathbb{P}_k^{n+1}$ that vanish on the variety mentioned above.

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