SDIT and Non-Commutative Ranks of Matrix Lie Algebras

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OUTLINE

1. **Introduction**

   Problem and motivation

2. **Our Results**

   Result overview with quick survey

3. **Lie Algebras**

   Definitions and complete statements of results

4. **Conclusion**

   Summary and open problems

1.

Introduction

Problem statement and motivation

Polynomial Identity Testing (PIT)

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$$f(x_1, \cdots, x_m)$$

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- Is it even in NP ? Succinct certificates?

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  - Kabanets—Impagliazzo '04 - SDIT $\in$ NSUBEXP implies circuit lower bounds.(NEXP $\not\subseteq$ P/poly or VP $\neq$ VNP)
  - One upshot is that we can now hope to use some linear algebra.

The matrix space viewpoint

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- Let $\mathscr{A}$ be the linear space spanned by the tuple of $n \times n$ matrices $(A_1, \cdots, A_m)$ over $\mathbb{F}$.

- $\mathscr{A}$ is singular if every matrix in it is singular i.e., if the symbolic determinant $\det\left(\sum_i x_i A_i\right)$ is identically zero.

Two kinds of certificates

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- Certificate 1 - Shrunk subspace

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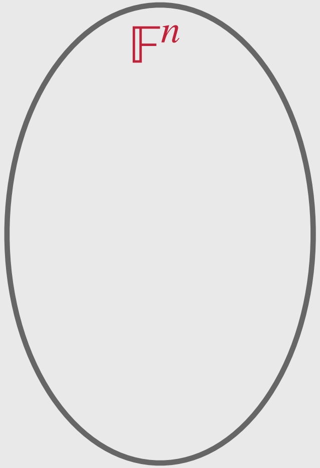

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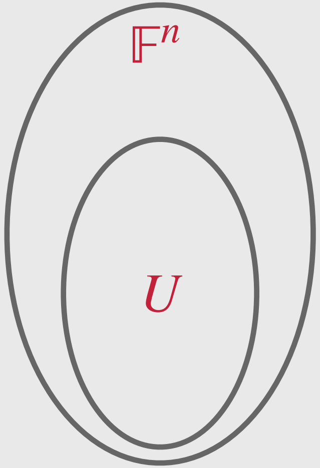

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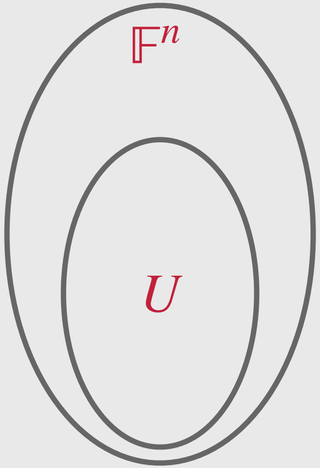

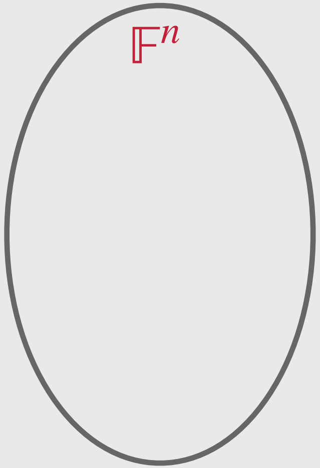

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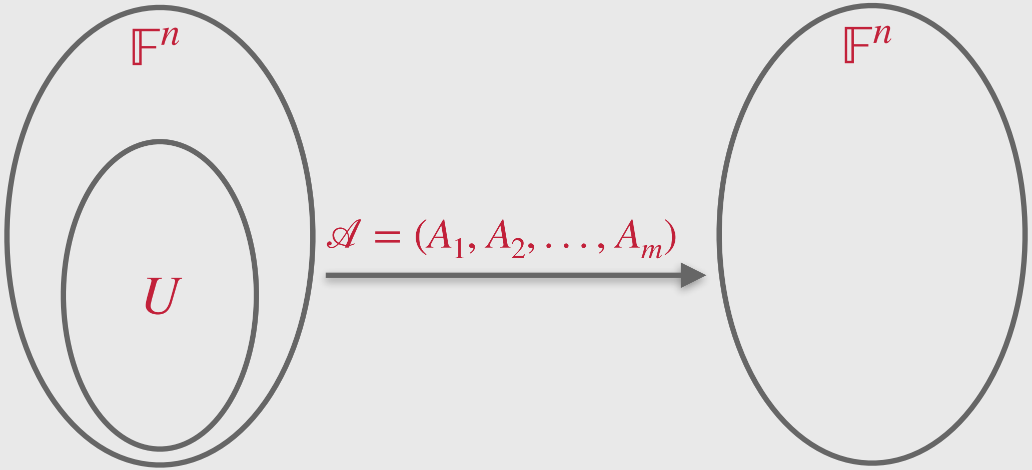

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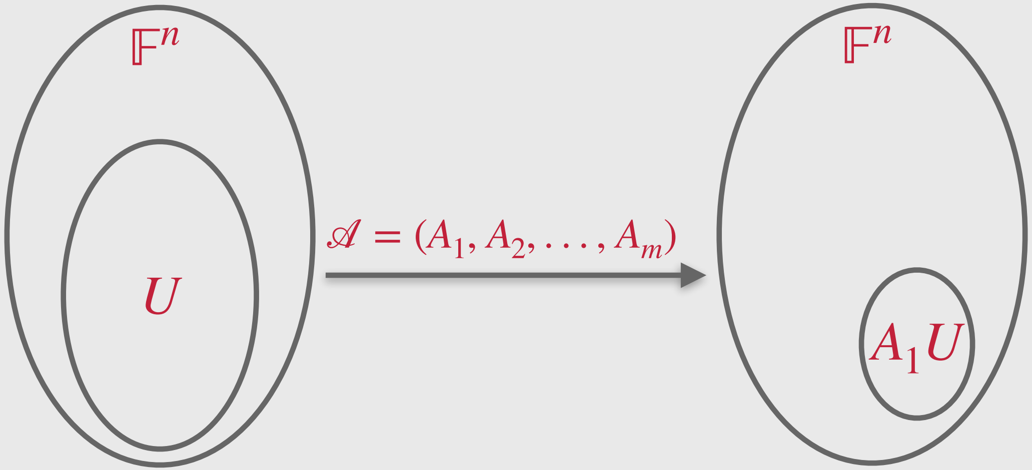

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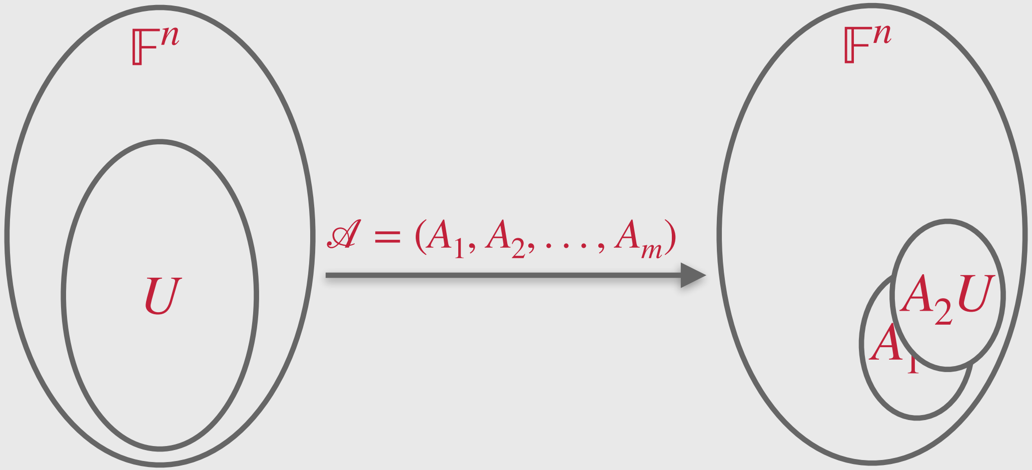

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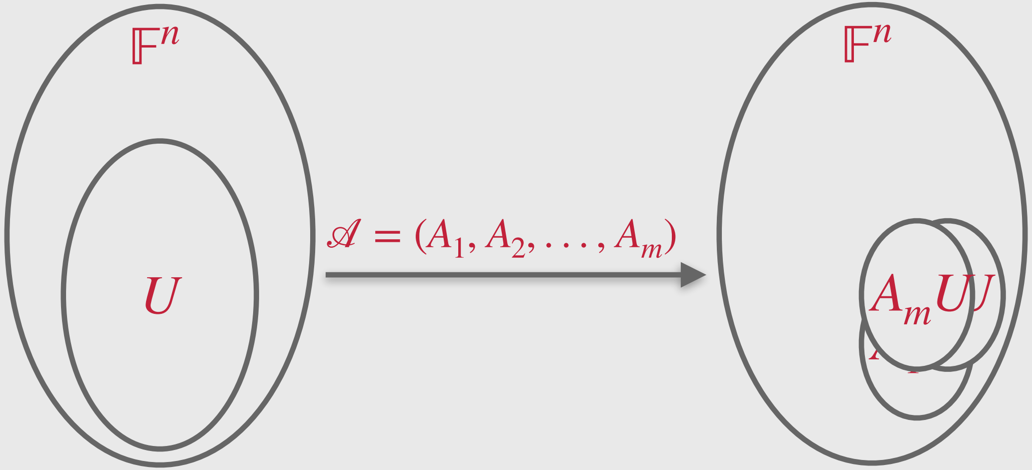

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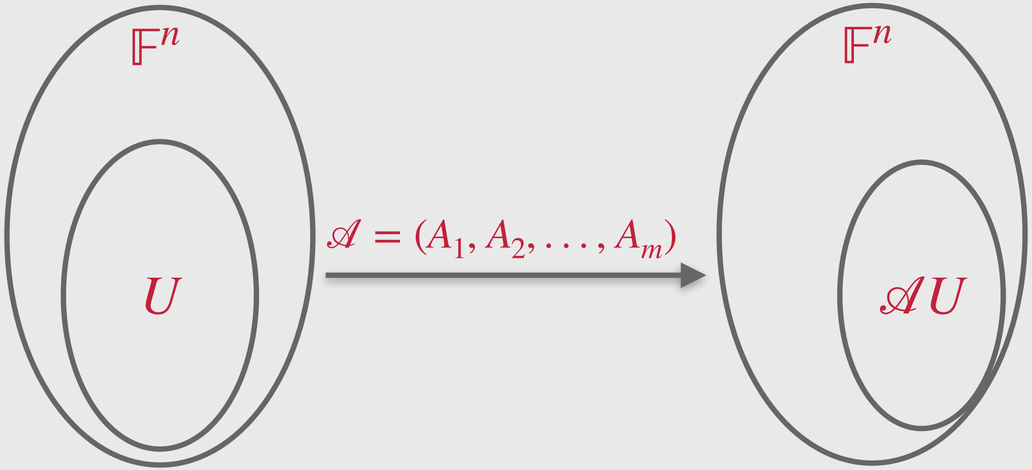

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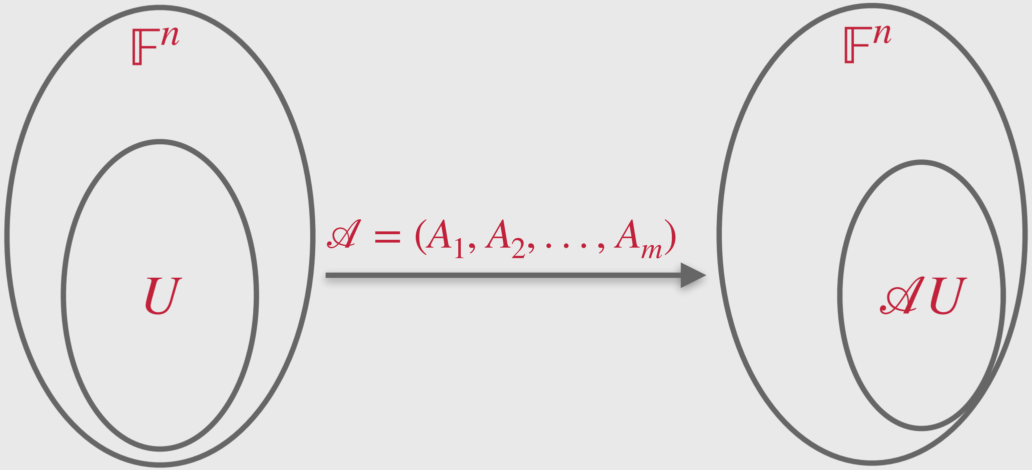
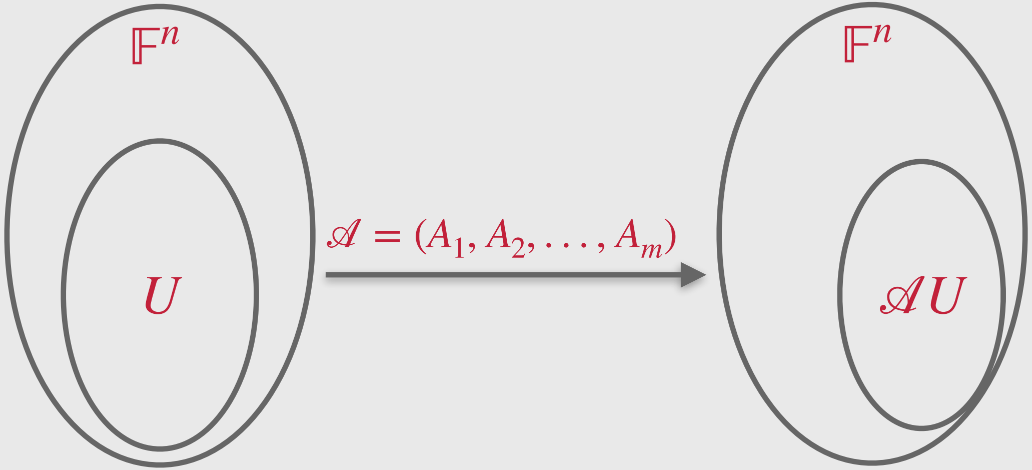


U is shrunk subspace if $\dim(\mathscr{A}U) < \dim(U)$

Two kinds of certificates

- Certificate 1 - Shrunk subspace
- Certificate 2 - Linear Kernel



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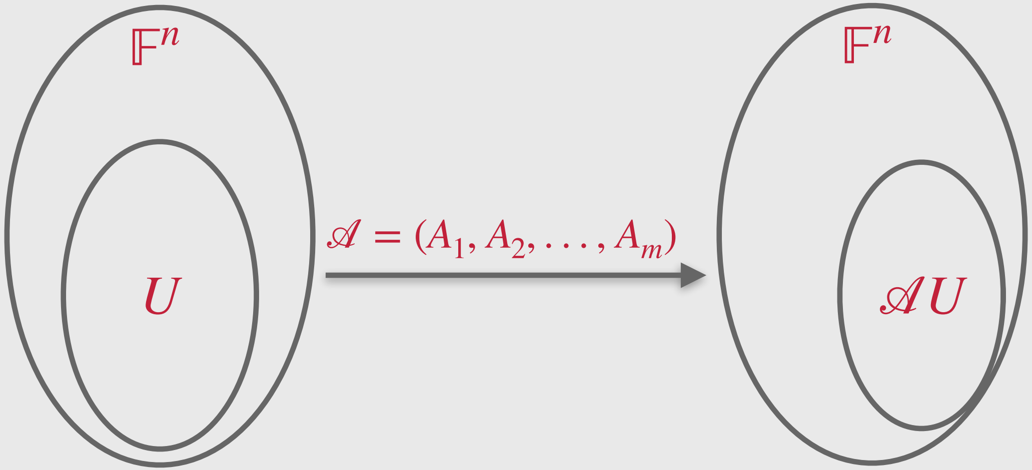

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- $(v_1, \cdots, v_m)$ is a linear kernel for $\mathscr{A}$

  if $\left(\sum_{i=1}^{m} x_i A_i\right) \sum_{i=1}^{m} x_i v_i = 0$

  identically.
- Such a certificate can be found and verified in $\mathrm{poly}(n)$ time.

A bit of history

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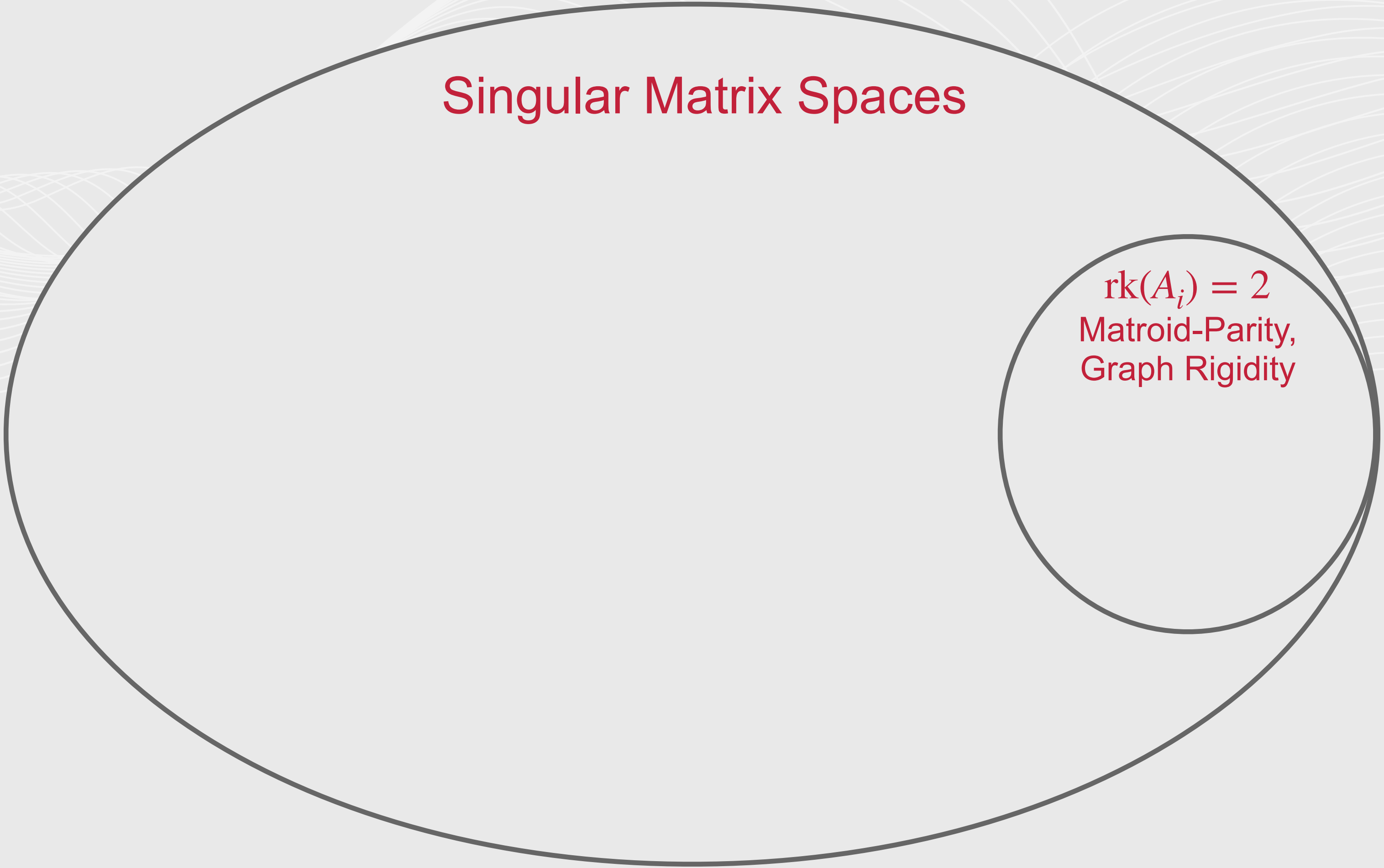
Singular Matrix Spaces

A bit of history



A bit of history

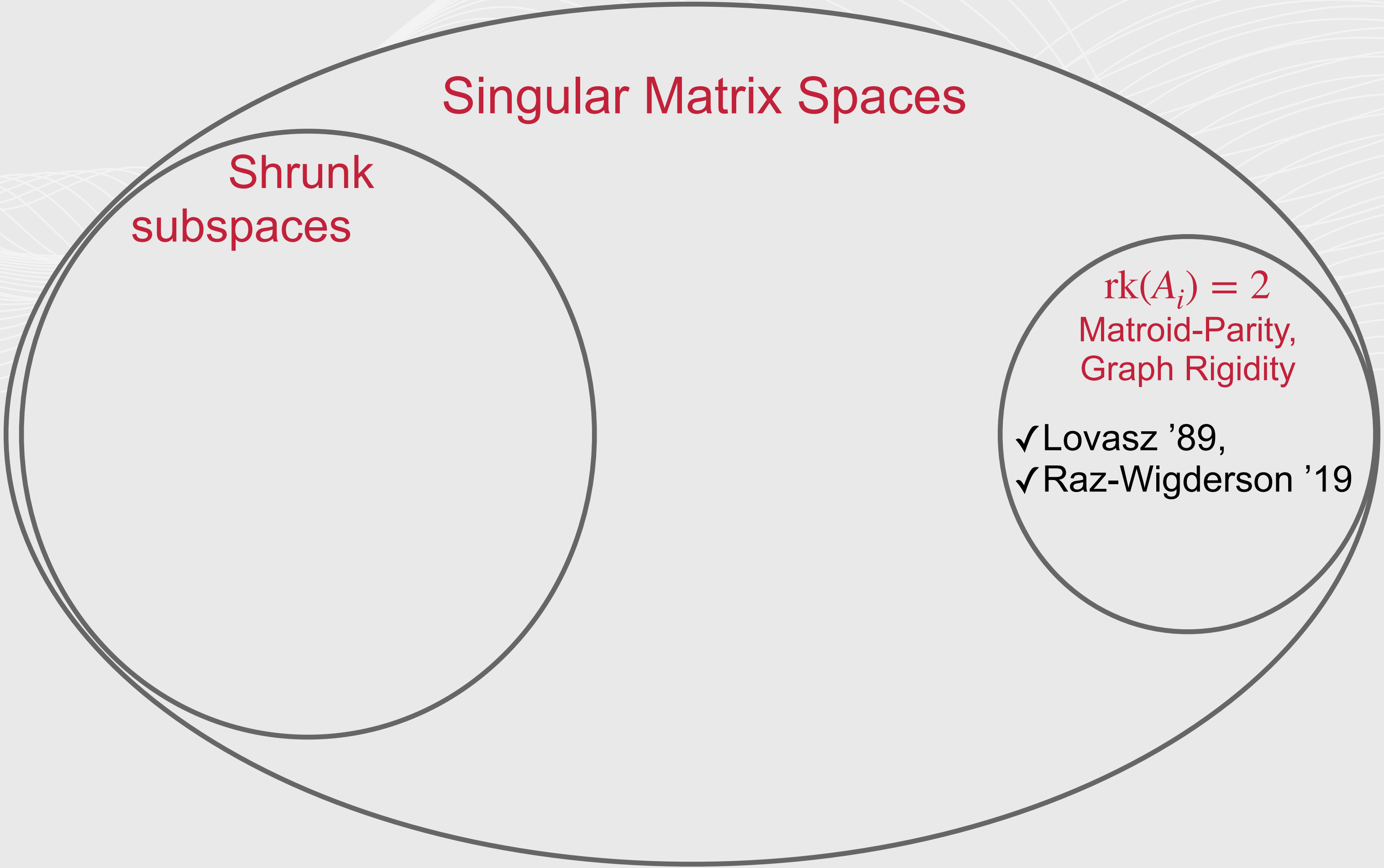
Singular Matrix Spaces

$\mathrm{rk}(A_i) = 2$

Matroid-Parity,
Graph Rigidity

✓Lovasz ’89,
✓Raz-Wigderson ’19

A bit of history



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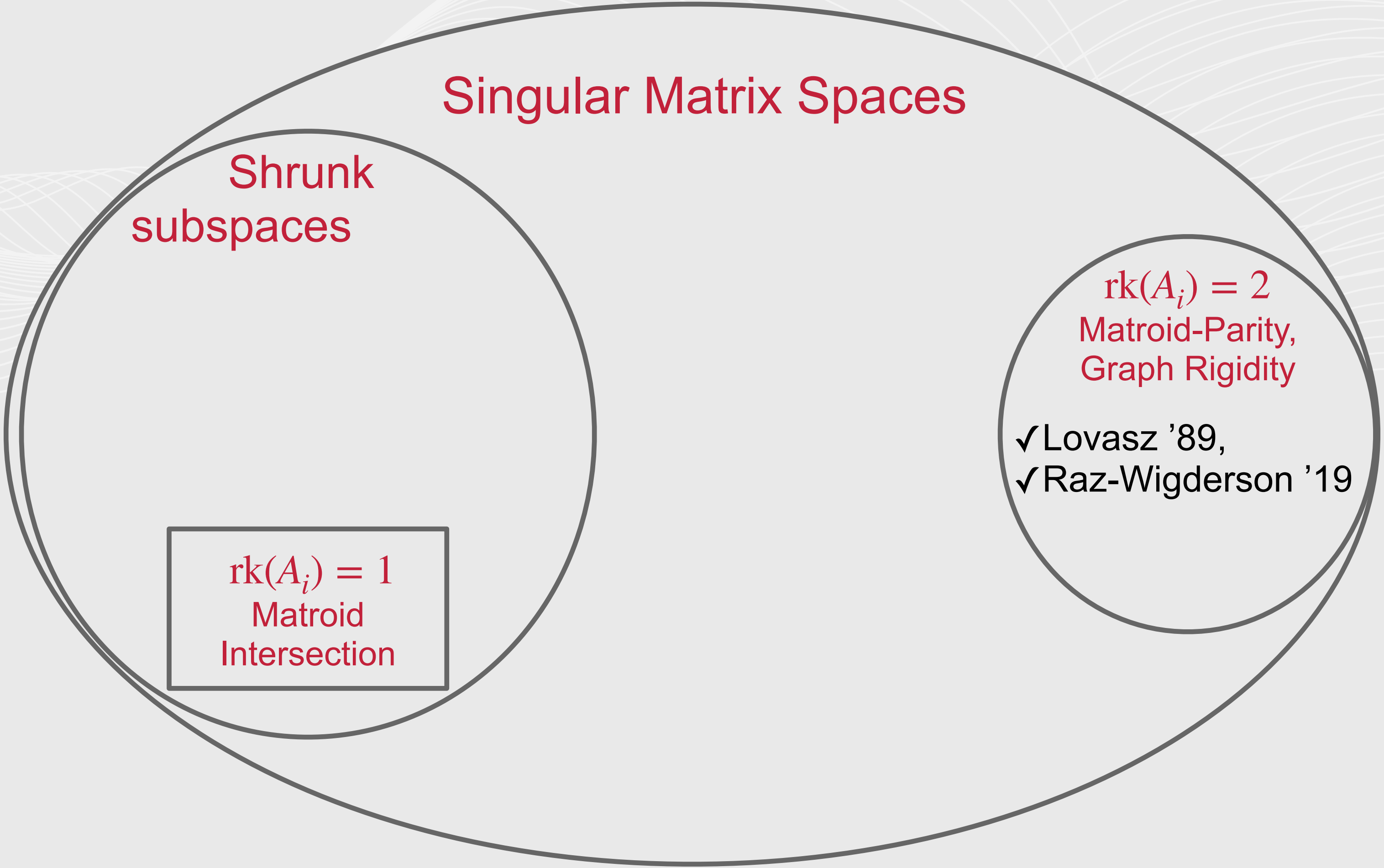

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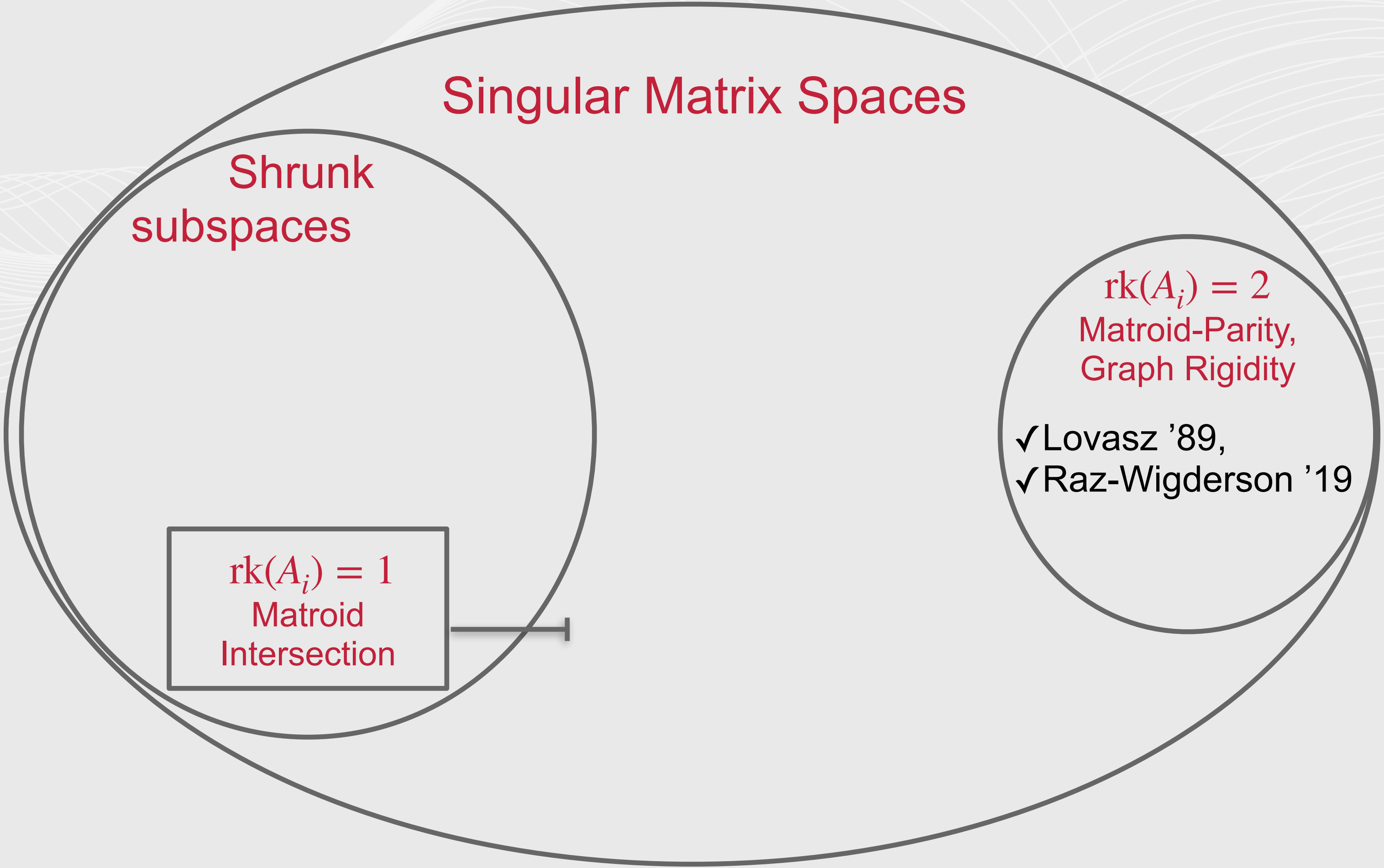

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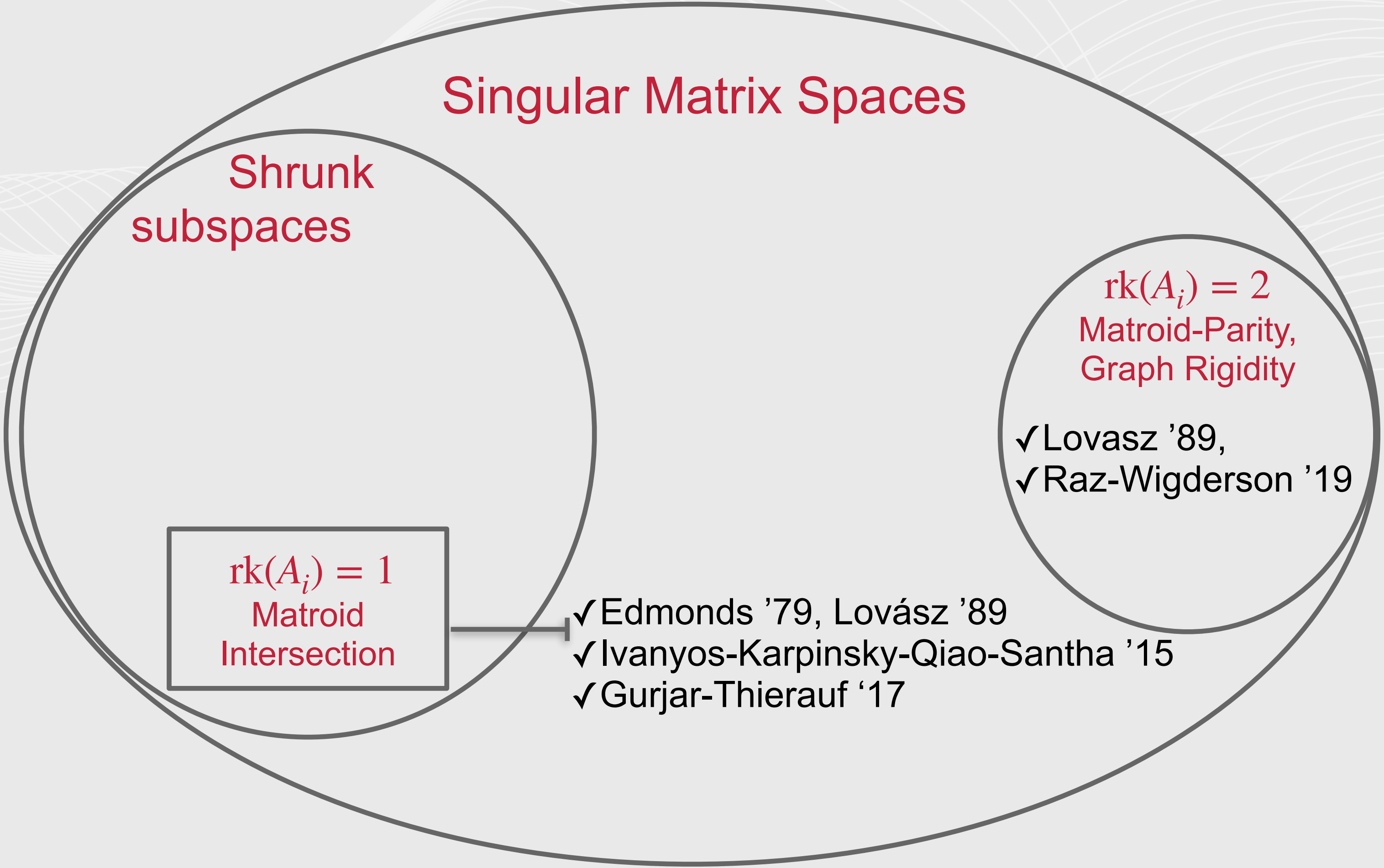

A bit of history

Singular Matrix Spaces

Shrunk subspaces

✓Garg-Gurvits-Oliveira -Wigderson ‘16

$\mathrm{rk}(A_i) = 1$

Matroid Intersection

✓Edmonds ’79, Lovász ’89
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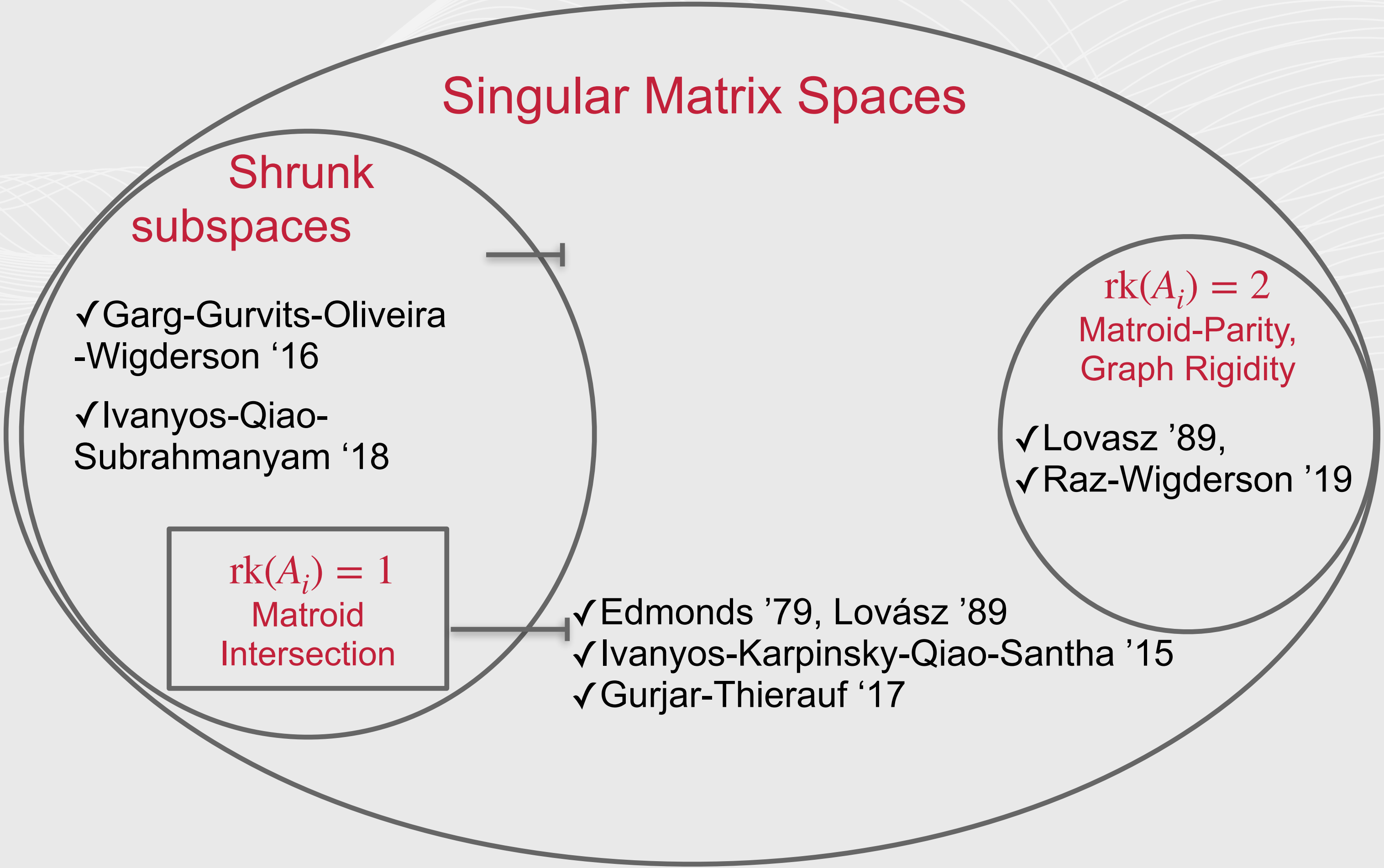

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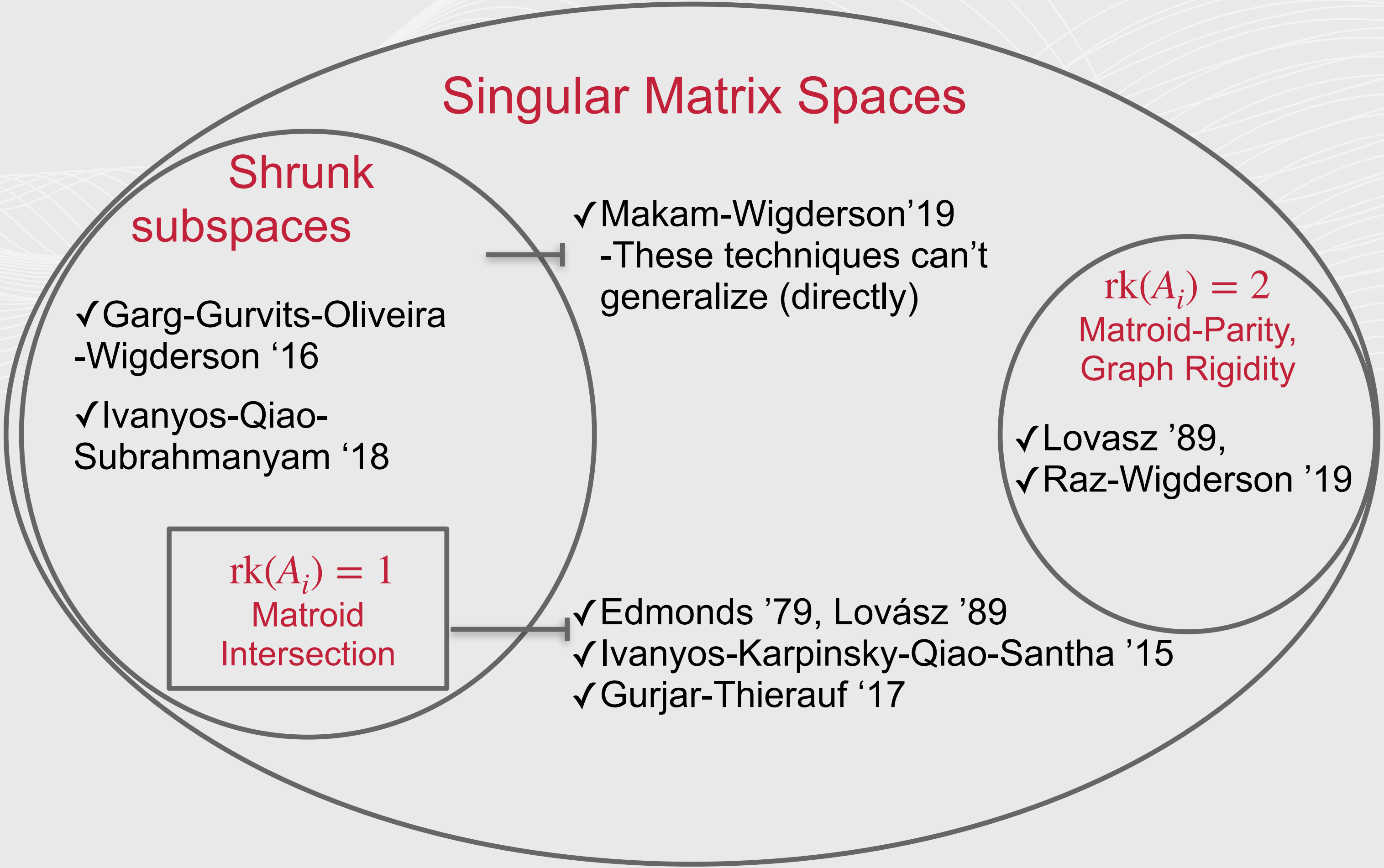

Singular matrix landscape

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Singular Matrix Spaces

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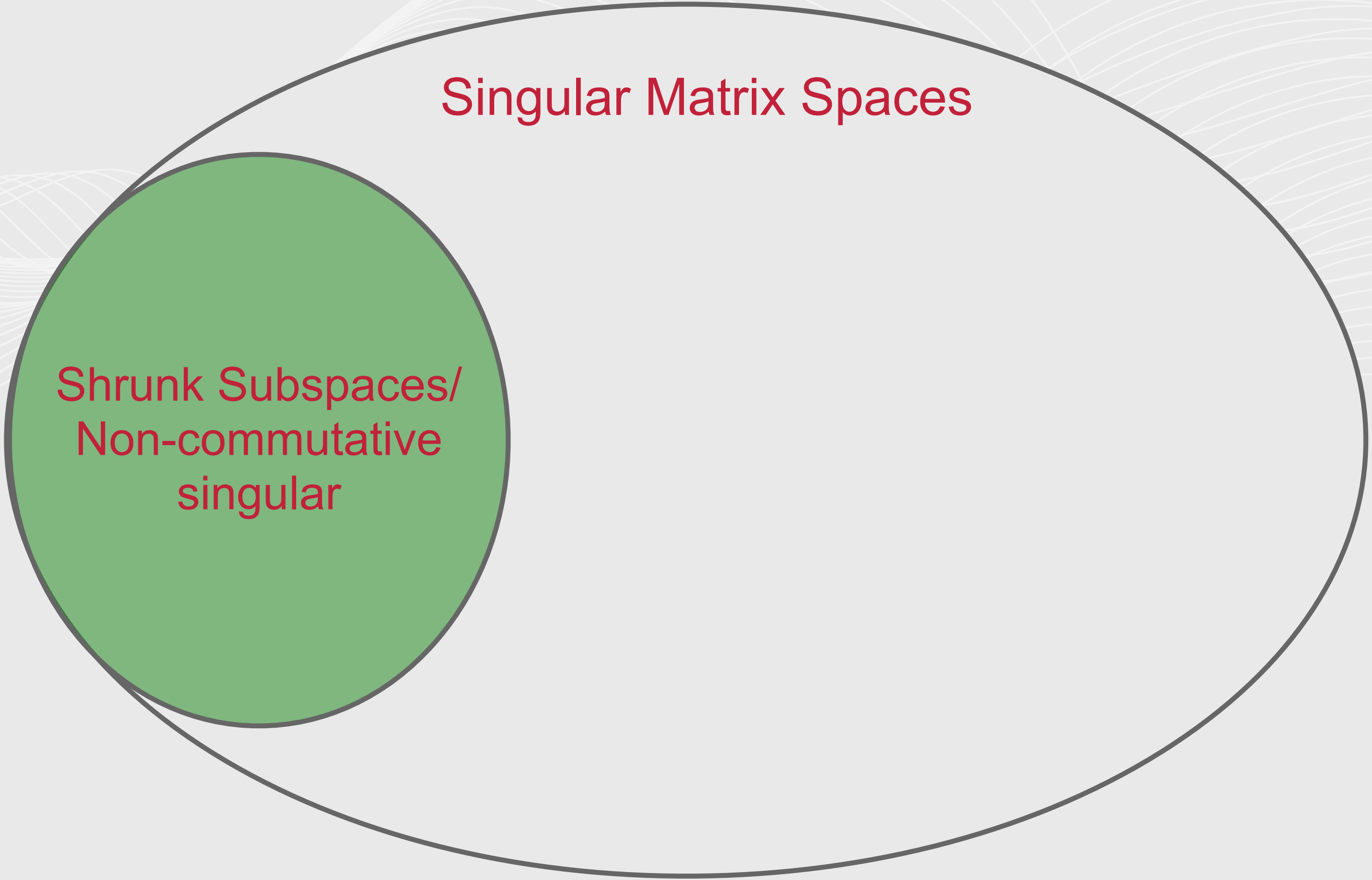

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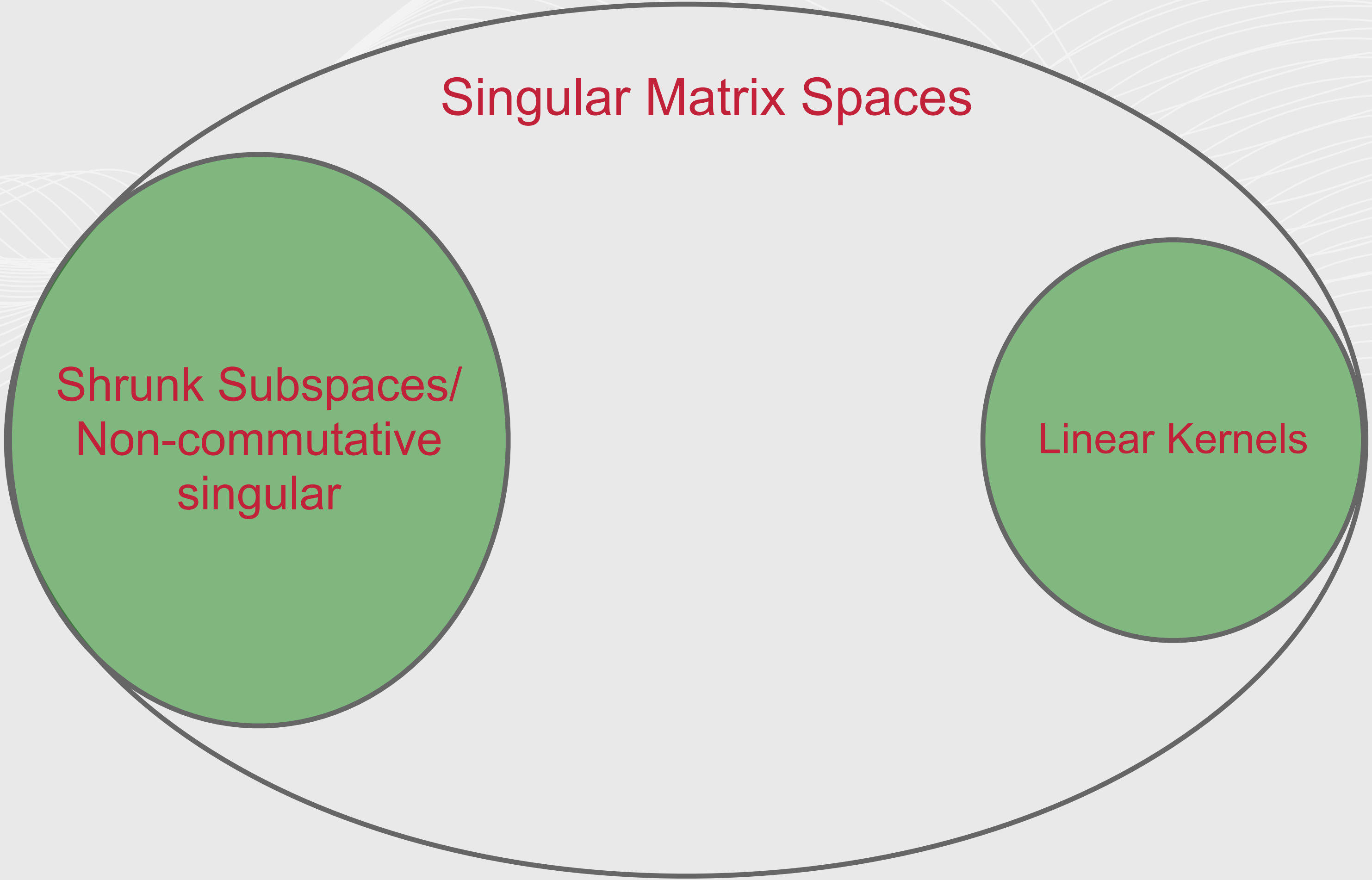

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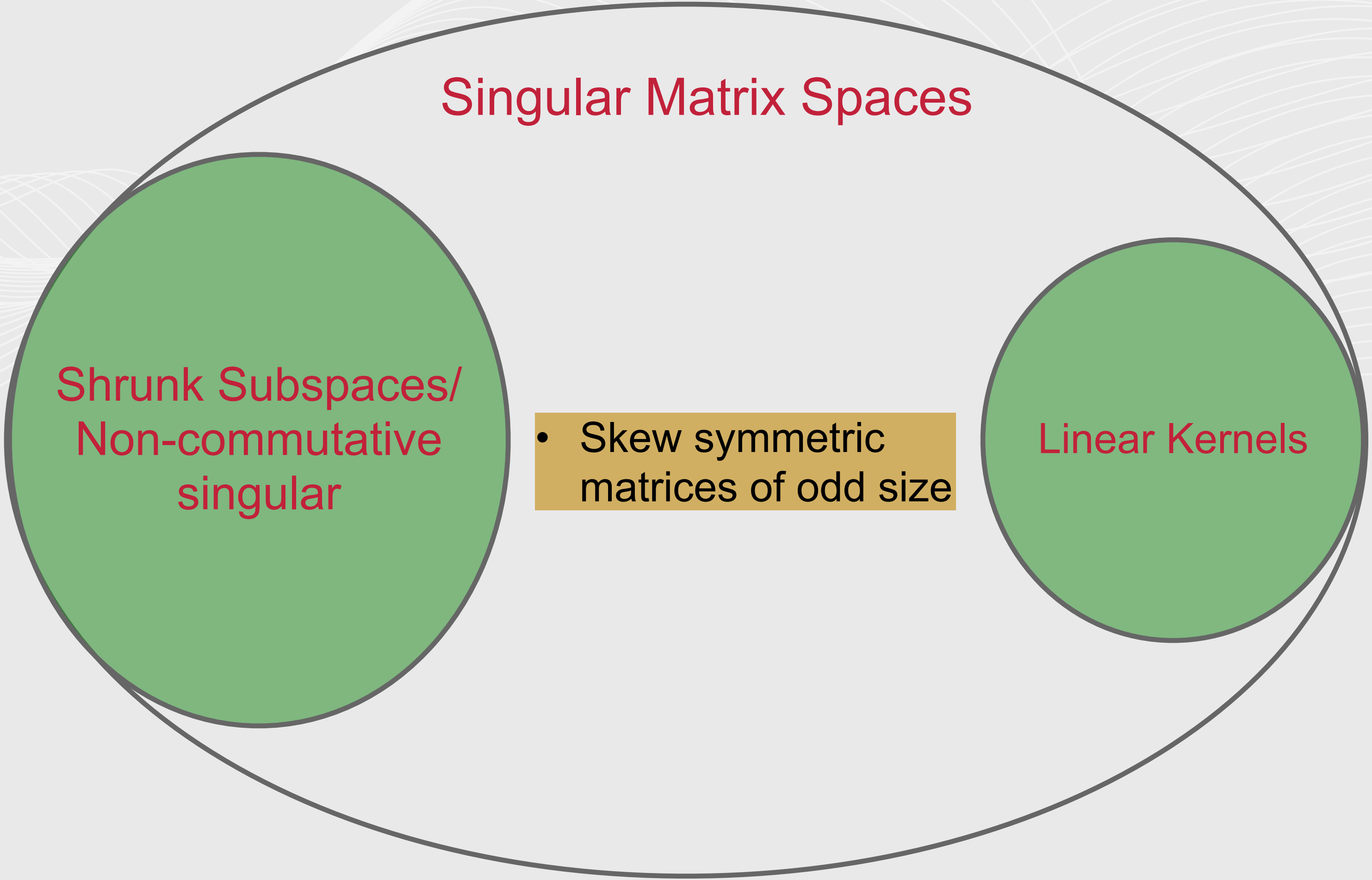

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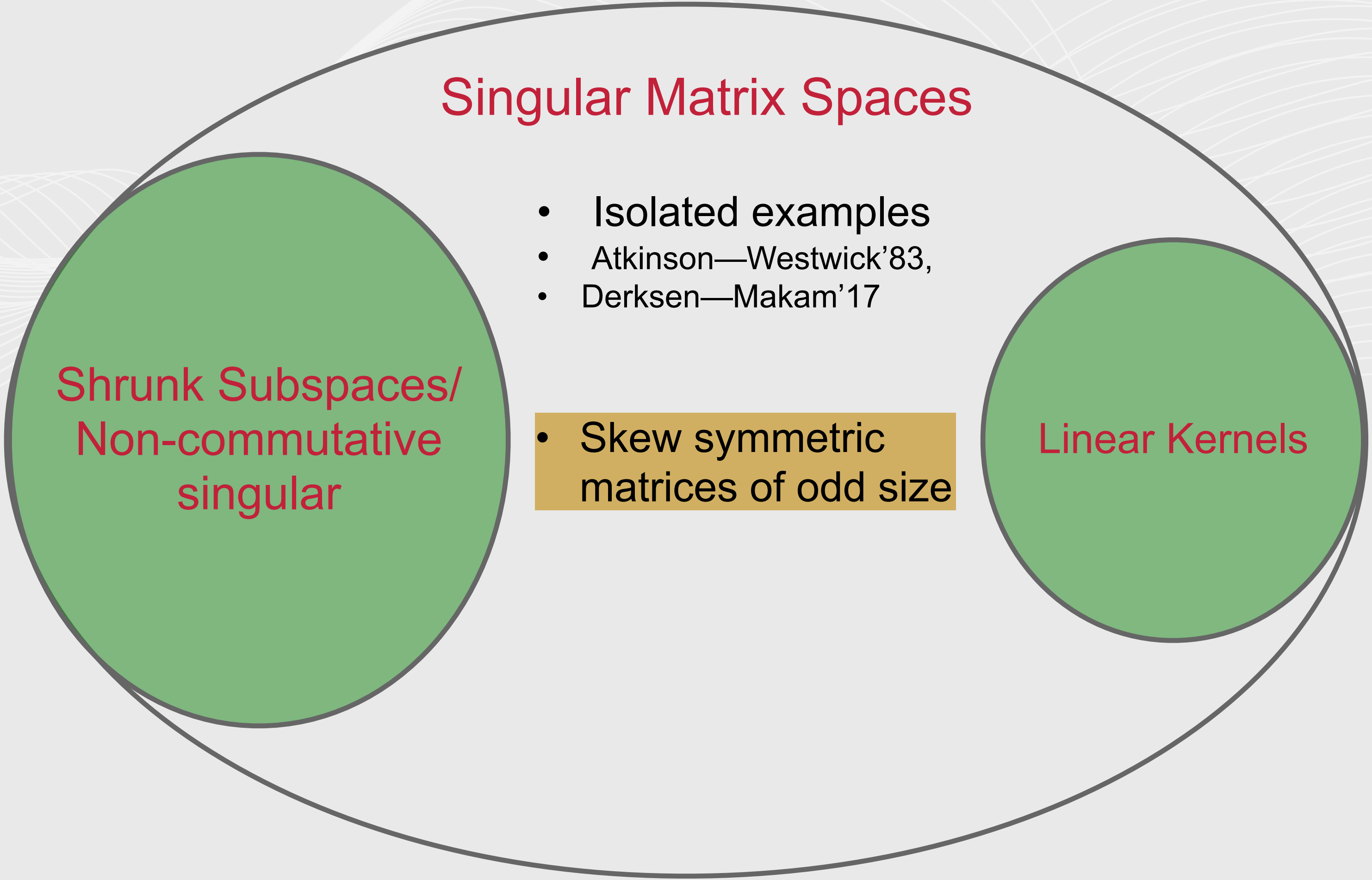

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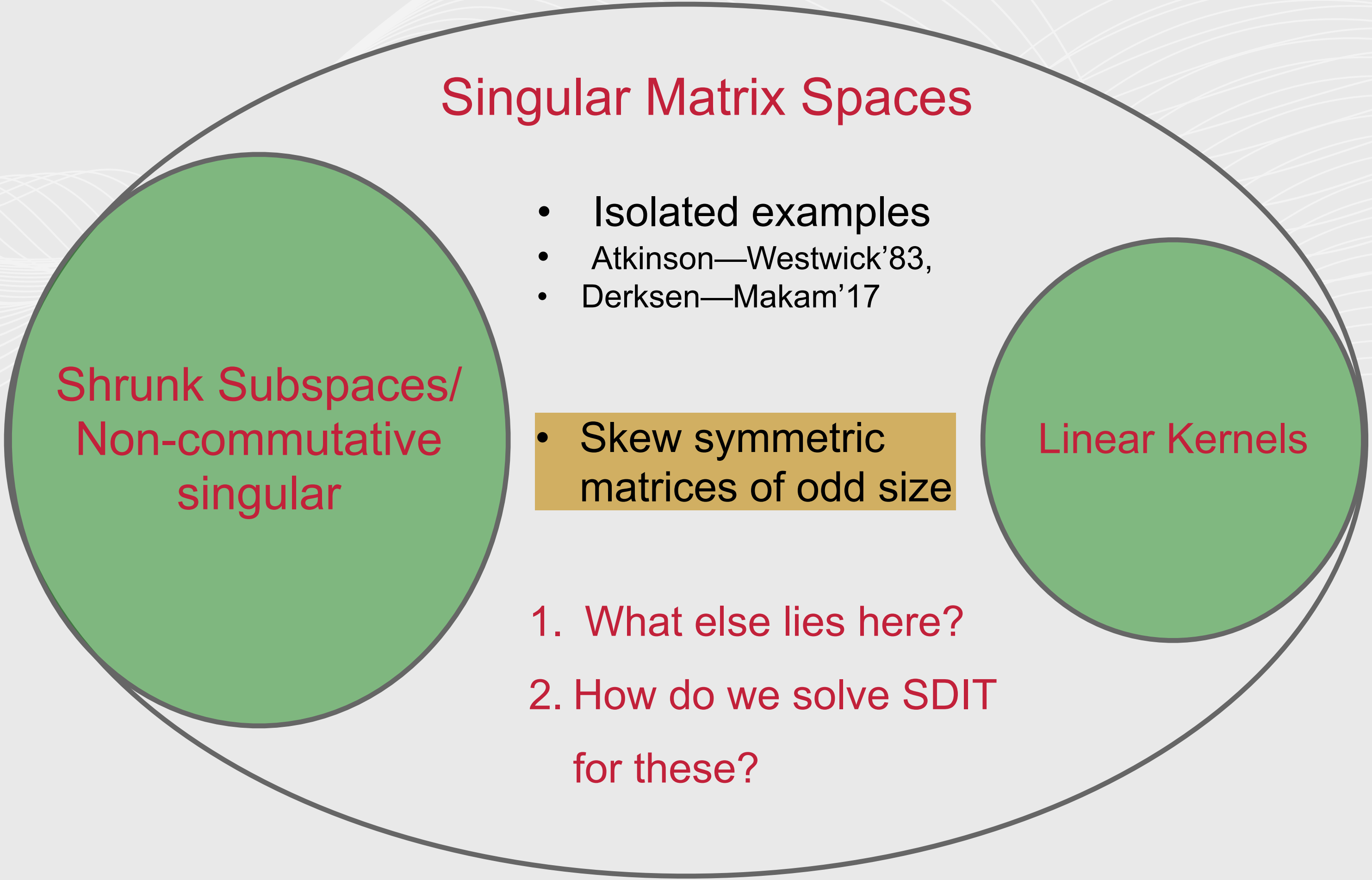

See Avi's talk at IAS titled "Linear Spaces of matrices" for more connections and background

Goal - To find more examples and solve SDIT for them

2.

Our Results

Overview of the results

Result 1- New examples

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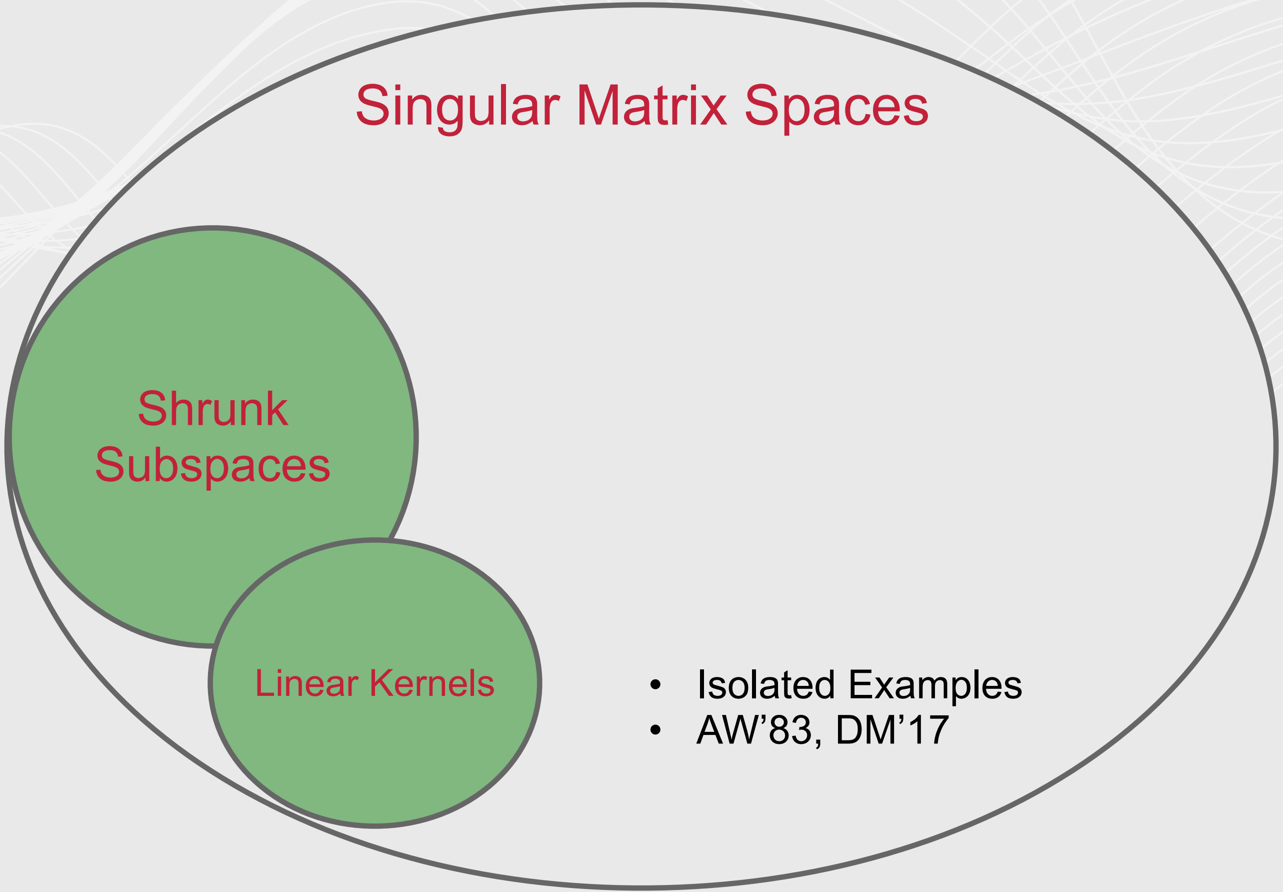

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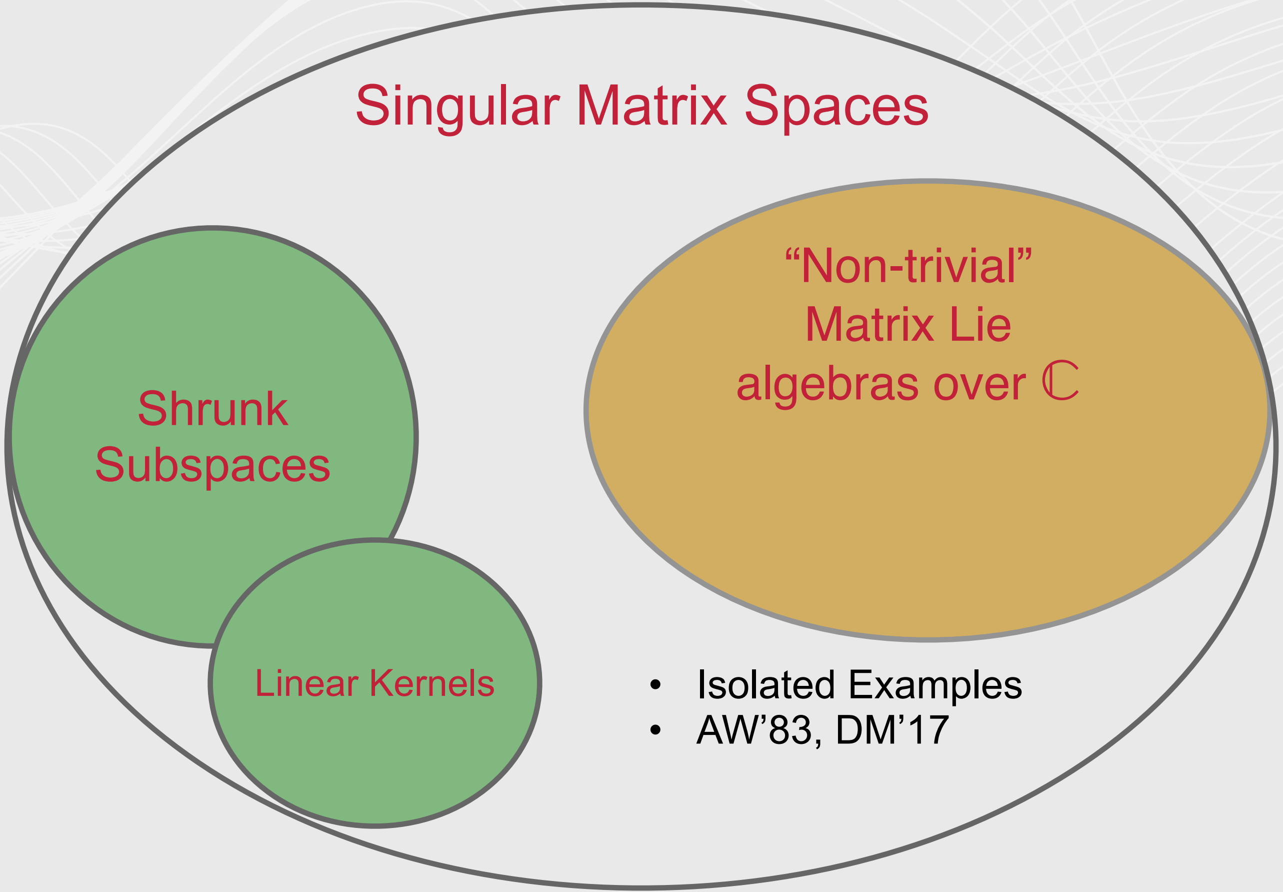

- Isolated Examples
- AW’83, DM’17

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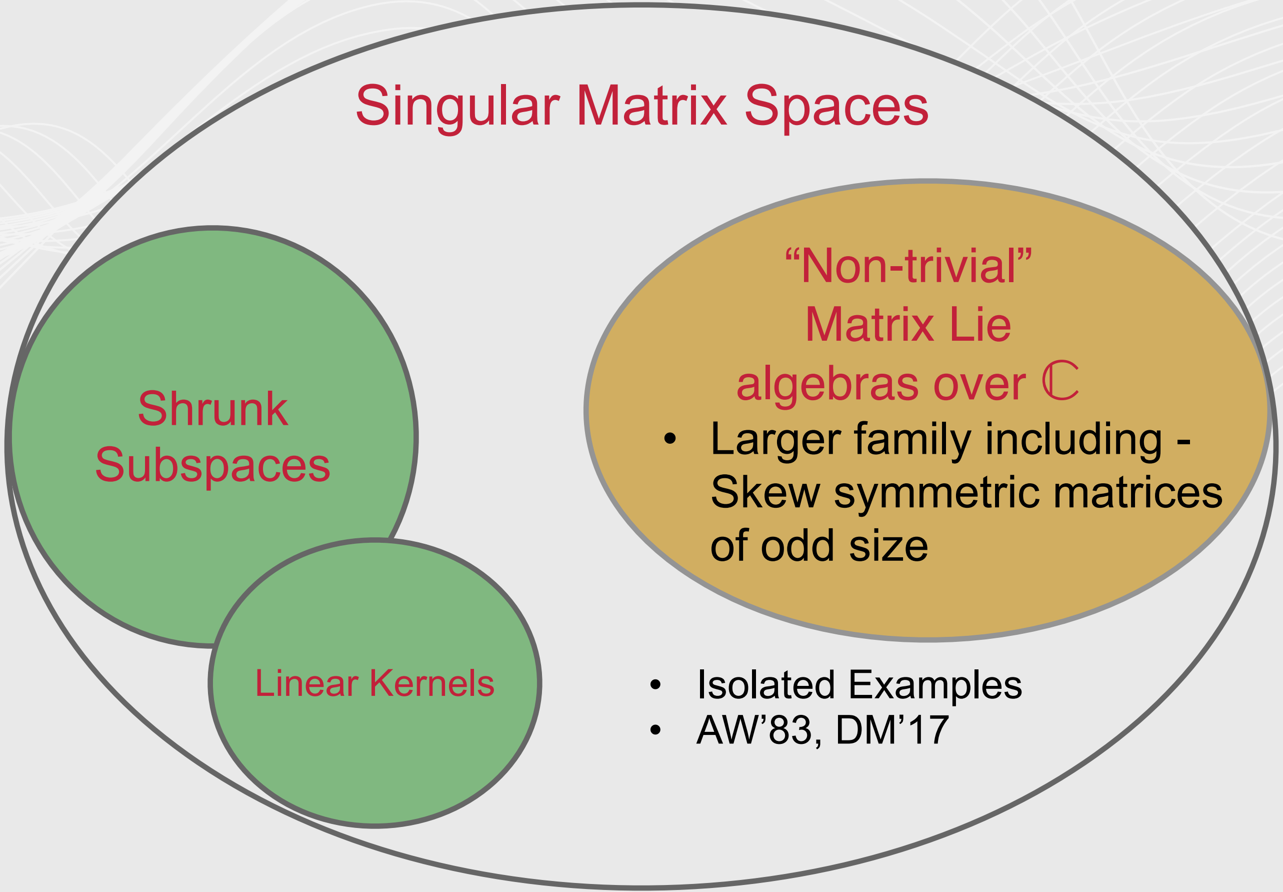

Result 2 - Algorithm

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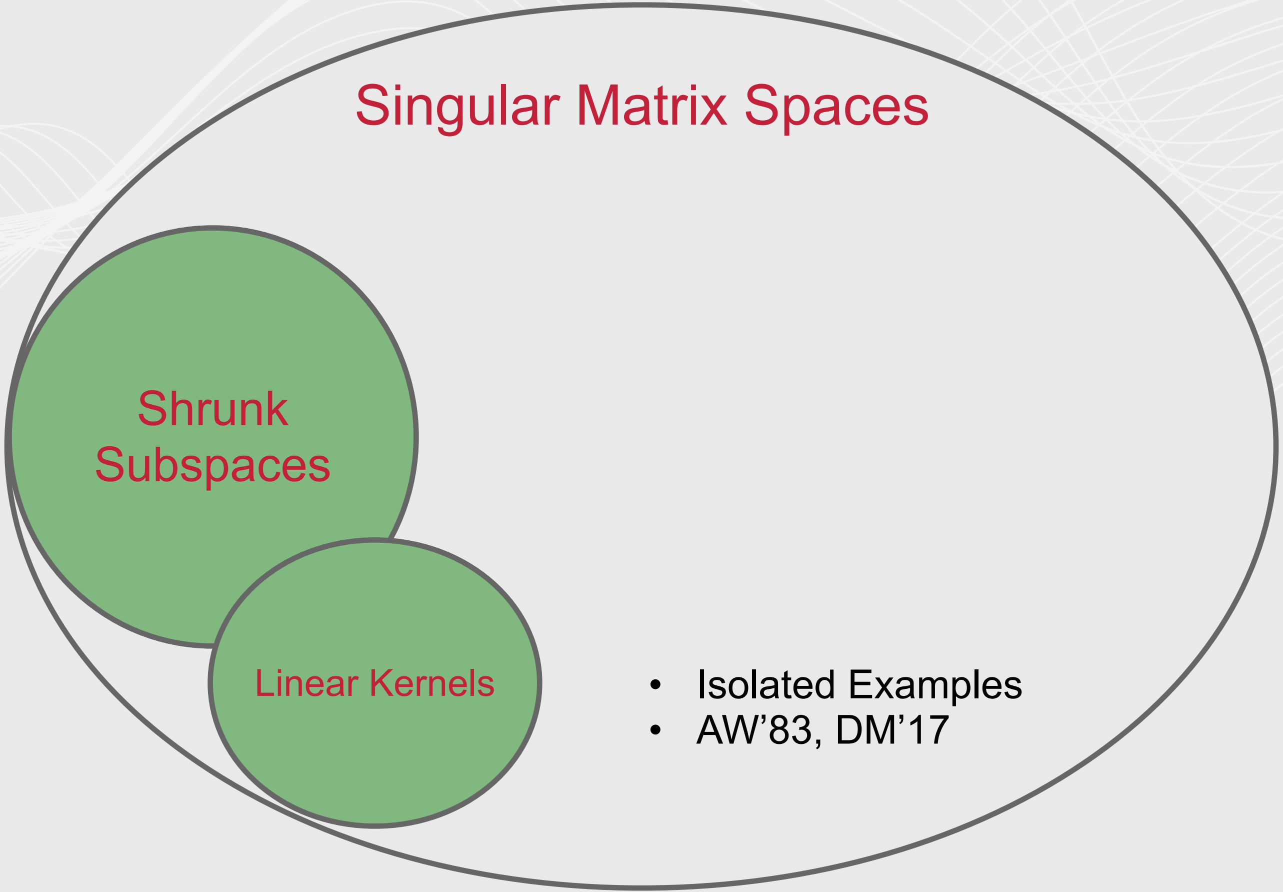

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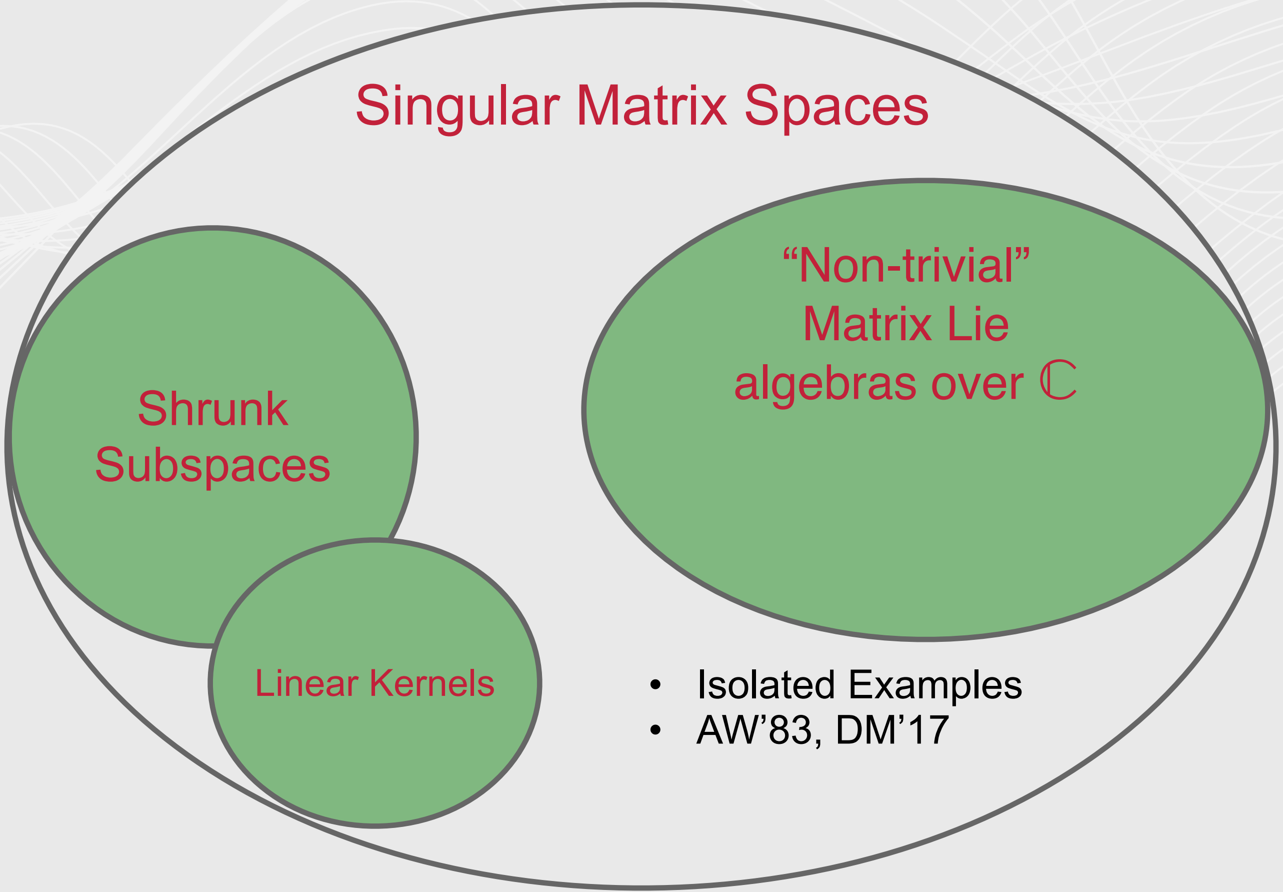

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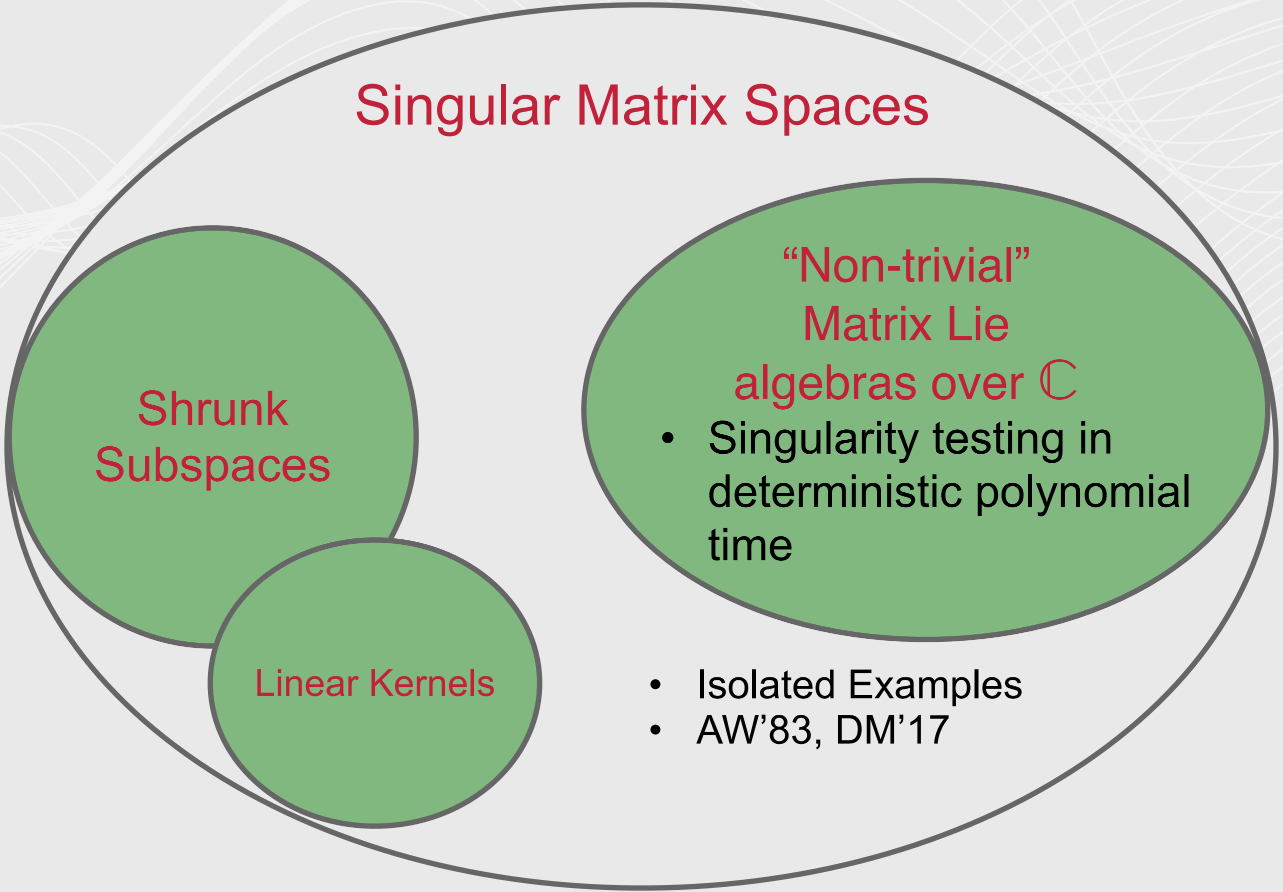

3.

Lie Algebras

Vector Spaces with a bracket

Matrix Lie algebras

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- A matrix Lie algebra $\mathscr{A}$ is irreducible if for every subspace $U \neq \{0\}, \mathbb{F}^n$ we have $\mathscr{A}U \nsubseteq U$.

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- This result also obtained independently by Makam-Derksen.

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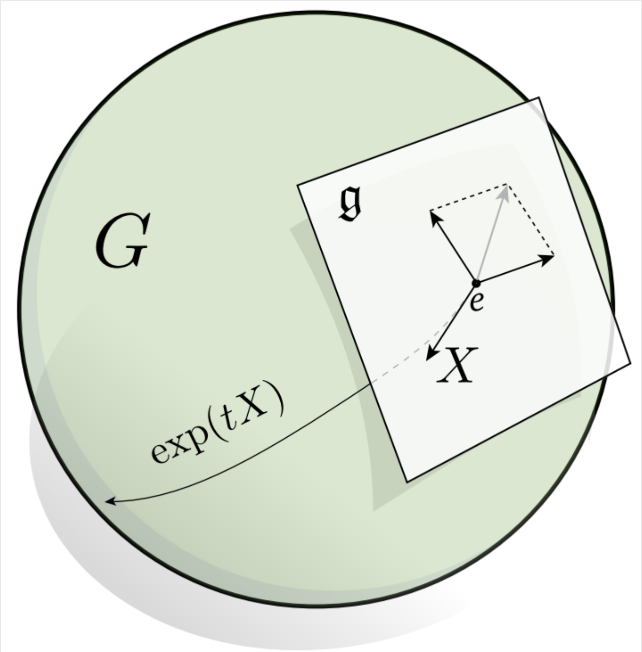


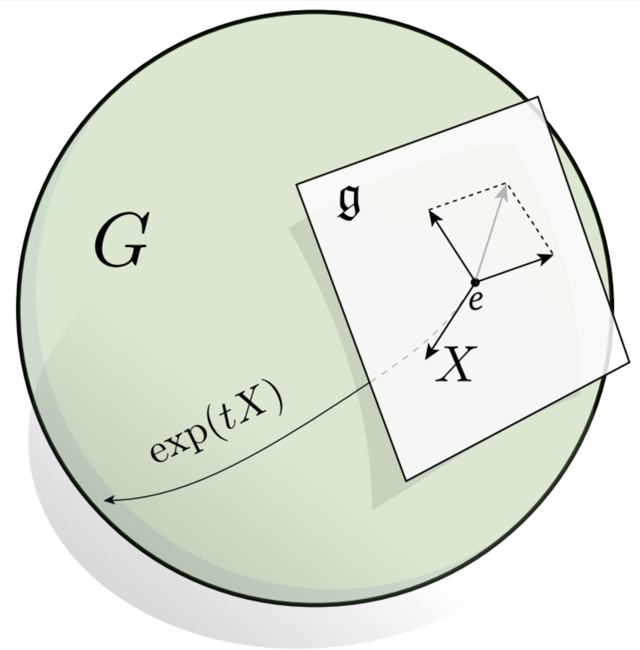
Timothy Budd, adapted from
Illustrating Geometry - Keenan

Lie group <—> Lie algebra

- Lie group - Group that is also a manifold.

- Lie algebras $\mathfrak{g}$ over $\mathbb{C}$ <-> Tangent spaces of a Lie groups G.

- Denoted as $\mathrm{Lie}(G) = \mathfrak{g}$.
  - Eg - $\mathrm{Lie}(\mathrm{SL}_n) = \mathfrak{sl}_n$



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- Lemma - If $\mathscr{A}$ is nc-singular, there exists a canonical subspace that is invariant under any X such that X stabilizes $\mathscr{A}$, i.e., $X\mathscr{A}X^{-1} = \mathscr{A}$.
  - Fact - G stabilizes $\mathrm{Lie}(G)$.

General version via composition factors

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- For every $\mathscr{A}$, there exists a *composition series* consisting of a finite set of irreducible matrix Lie algebras each called a *composition factor* of $\mathscr{A}$.

- Theorem 1 - Let $\mathscr{A}$ be a matrix Lie algebra over $\mathbb{C}$ such that none of the composition factors are zero. Then, $\mathscr{A}$ does not have shrunk subspaces.

Theorem 2

Note - Here we assume (wlog) that ρ is faithful and the statement is true for any algebraically closed field F over characteristic not 2 or 3.

Theorem 2

- Theorem 2 - Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbb{C}$ and ρ be an irreducible representation of $\mathfrak{g}$. Then, $\mathscr{A} = \rho(\mathfrak{g})$ admits a linear kernel if and only if

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  - $\rho = 0$ or $\rho \cong \mathrm{ad}$.

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How do we build matrix Lie algebras?

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- Can build matrix lie algebras via a map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ such that $\rho[x, y] = [\rho(x), \rho(y)]$.

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  - If $\mathfrak{g}$ is a matrix Lie algebra, can take $\rho = \mathrm{id}$.

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- Every semi-simple Lie algebra has a root structure and admits a Chevalley basis.

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- Simple fact - SDIT for an upper-triangular space is PIT for product of linear forms, i.e., depth-2 circuit (easy) :)

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- de Graaf, Ivanyos, Rónyai '96 gave a deterministic algorithm to compute such a $\mathfrak{h}$.

4.

Conclusion

Summary and open problems

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- We also show that such spaces do not posses a simple certificate in the form of linear kernels. (ruling out certain brute-force linear algebraic algorithms)
- We give a deterministic algorithm to decide singularity of such spaces.

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Thank you!

- Glad to hear your feedback/questions - tushant@uchicago.edu.