Quantum codes from classical tools:

A survey

Tushant Mittal

Acknowledgements

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    - "*Not the worst talk I've seen*"- One of them.

OUTLINE

1. **Introduction**

   Overview and the classical world

2. **The Quantum World**

   Generalizing to quantum

2a. **Tensor Product**

   Definition and tensor product based constructions

2b. **Symmetry**

   Using symmetry to boost parameters

1.

Introduction

Overview
Classical Construction

Error-Correcting Codes

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(k-bits)

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Noisy Channel/
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Quantum

$|\psi\rangle \in \mathbb{C}^{2^k}$

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- Good codes - Infinite family $\{C_n\}$ such that $k, d = \Theta(n)$

The good codes landscape

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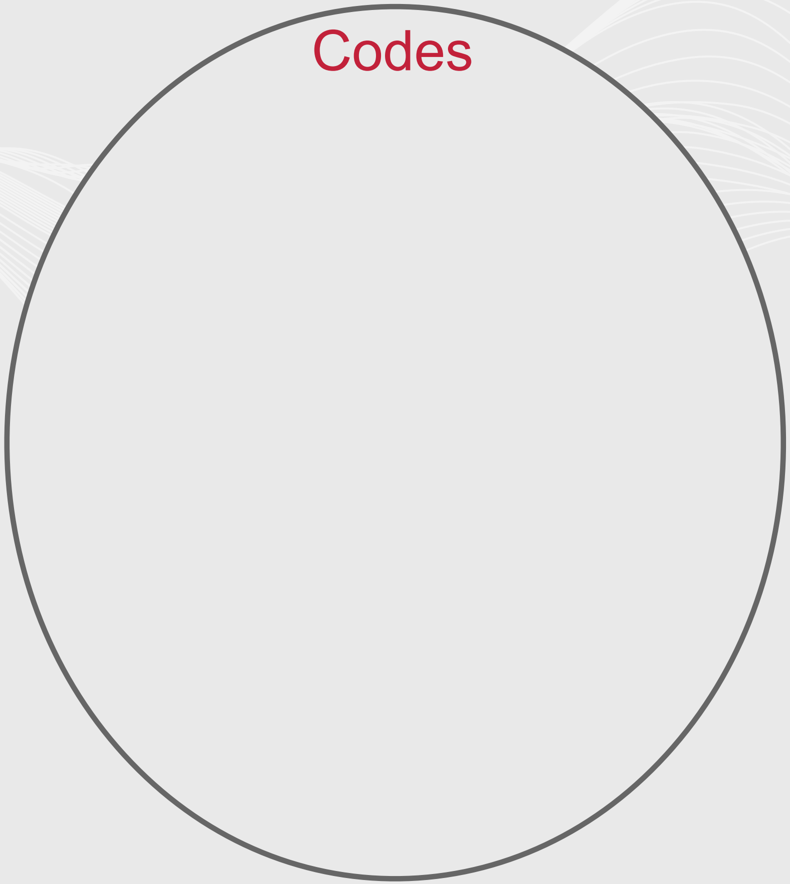

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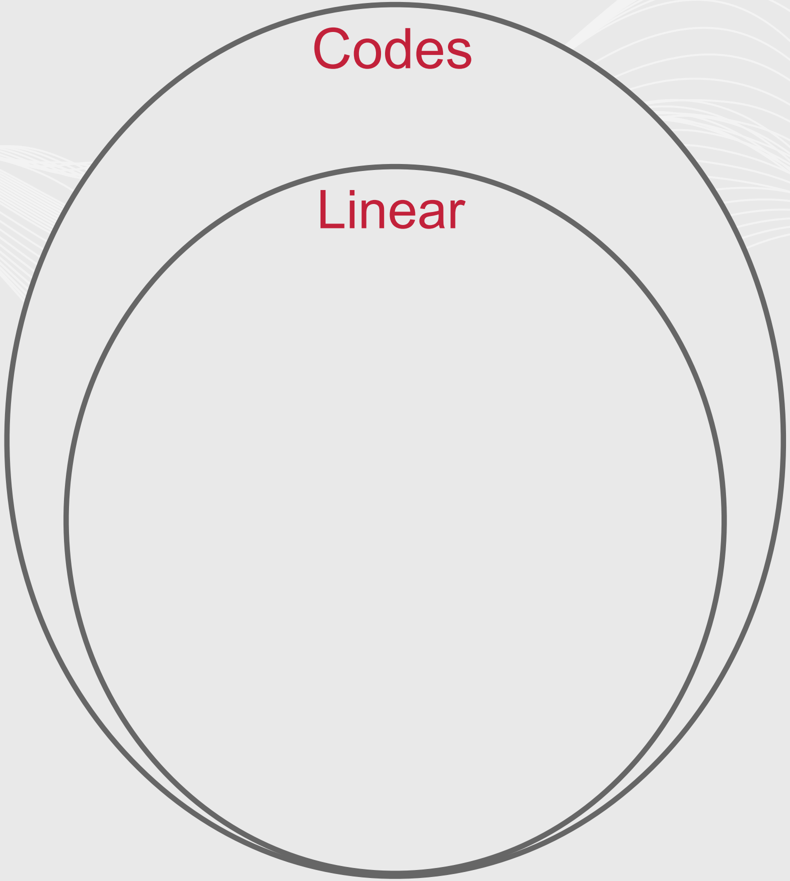

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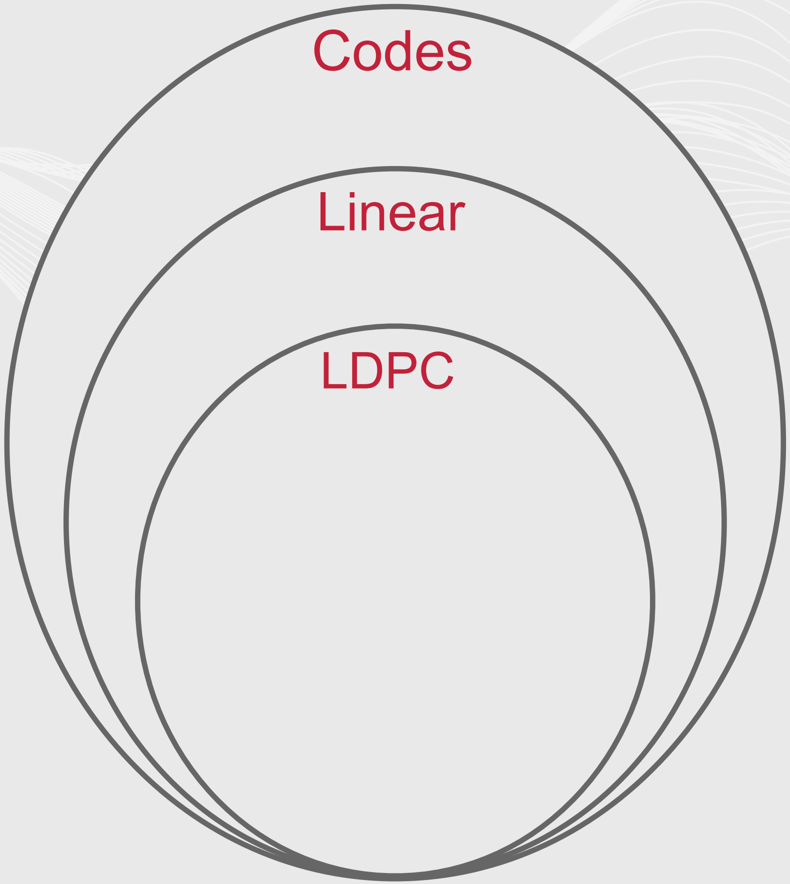

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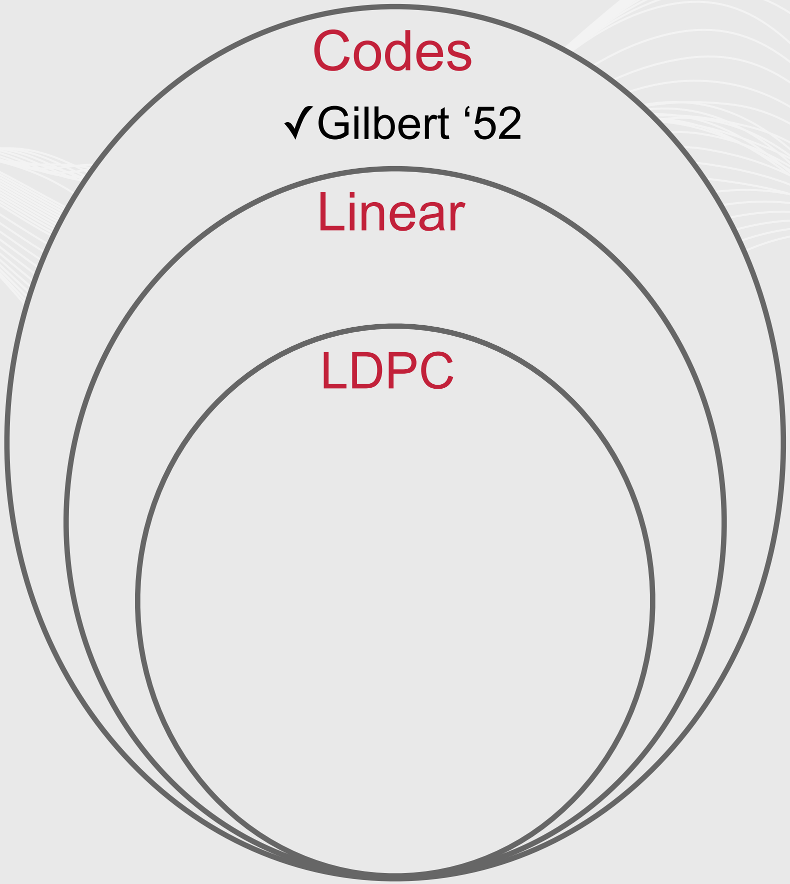

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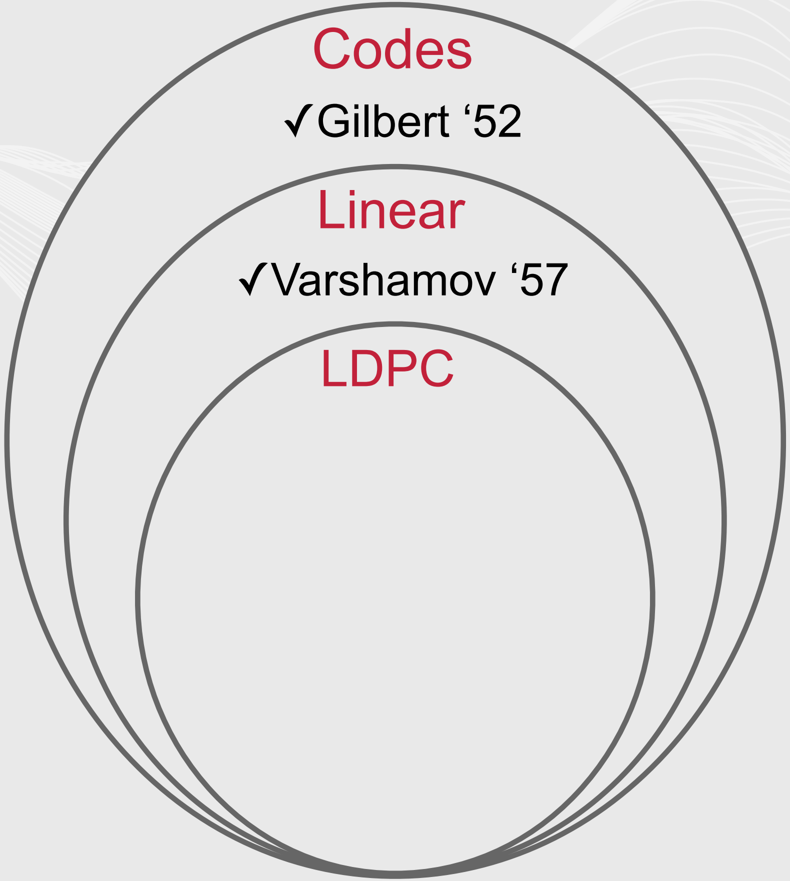

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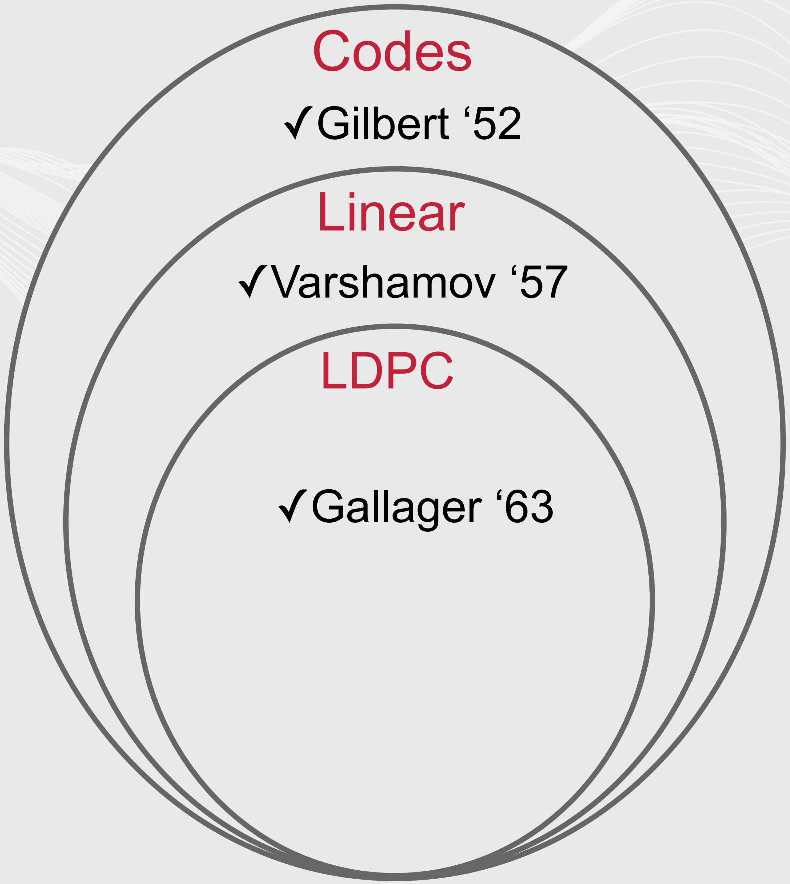

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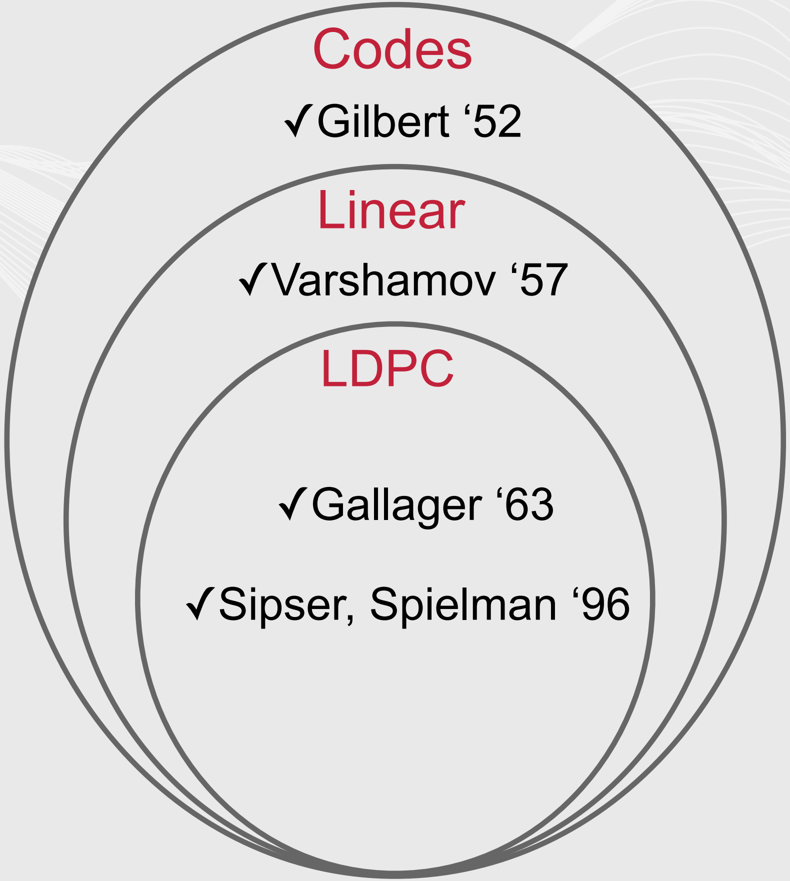

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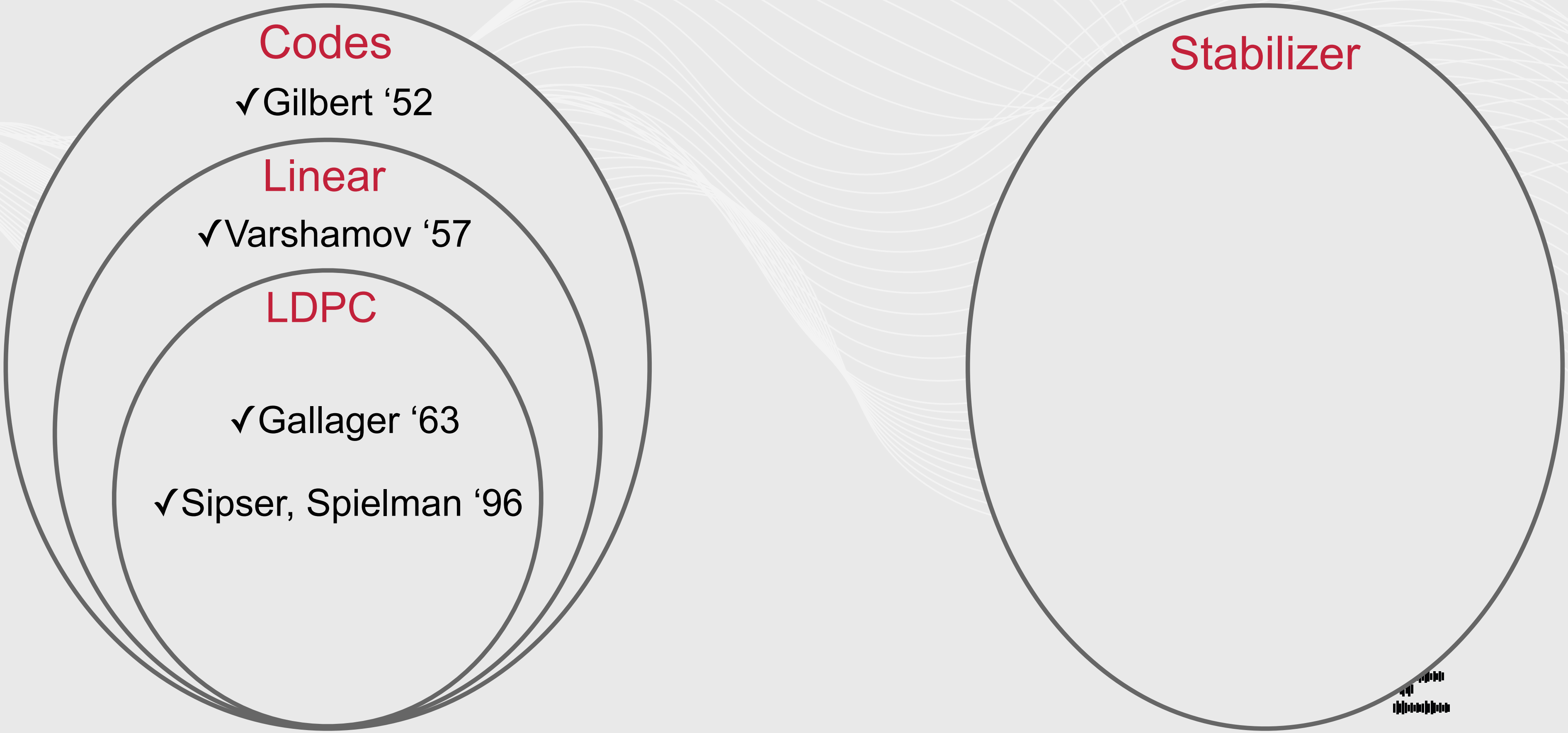

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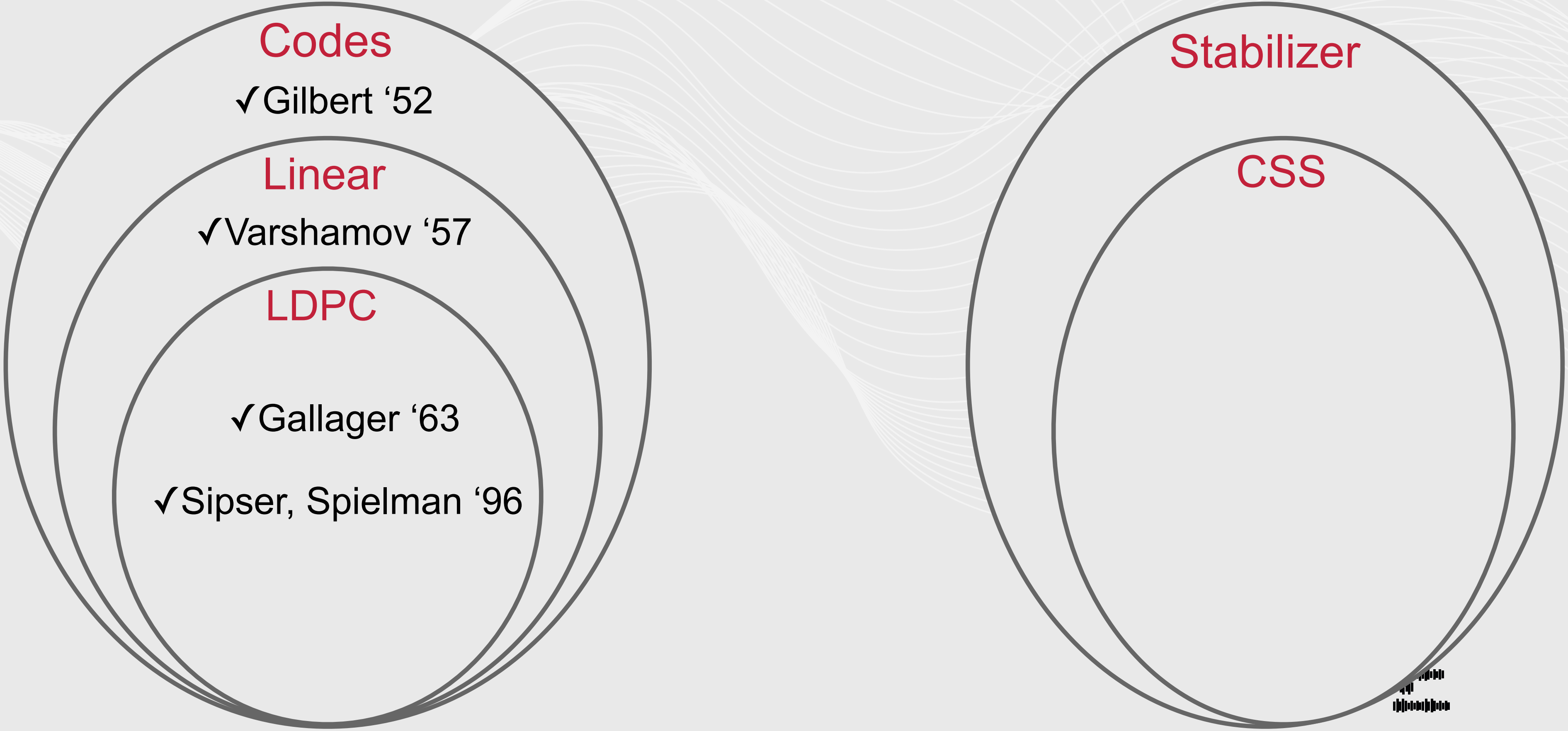

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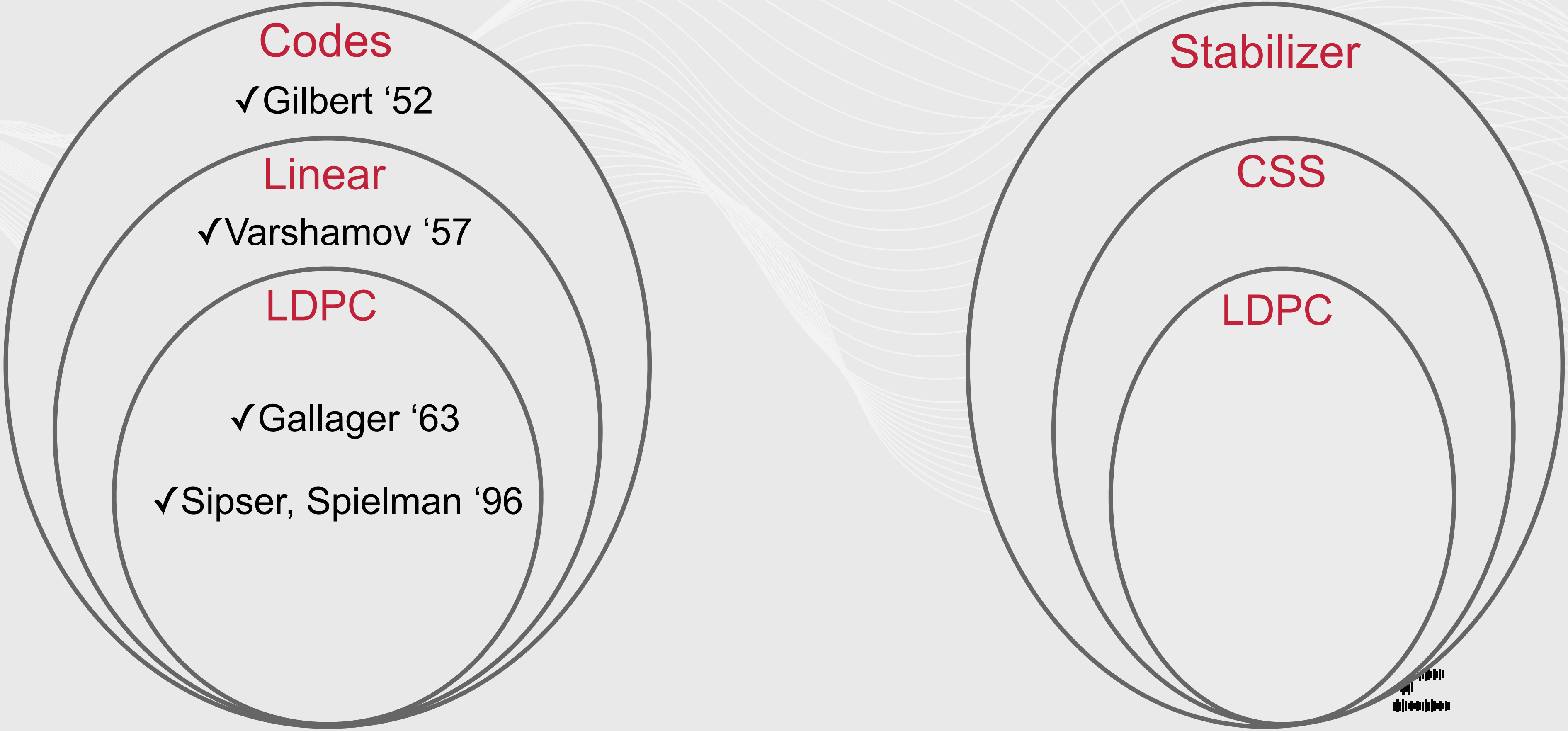

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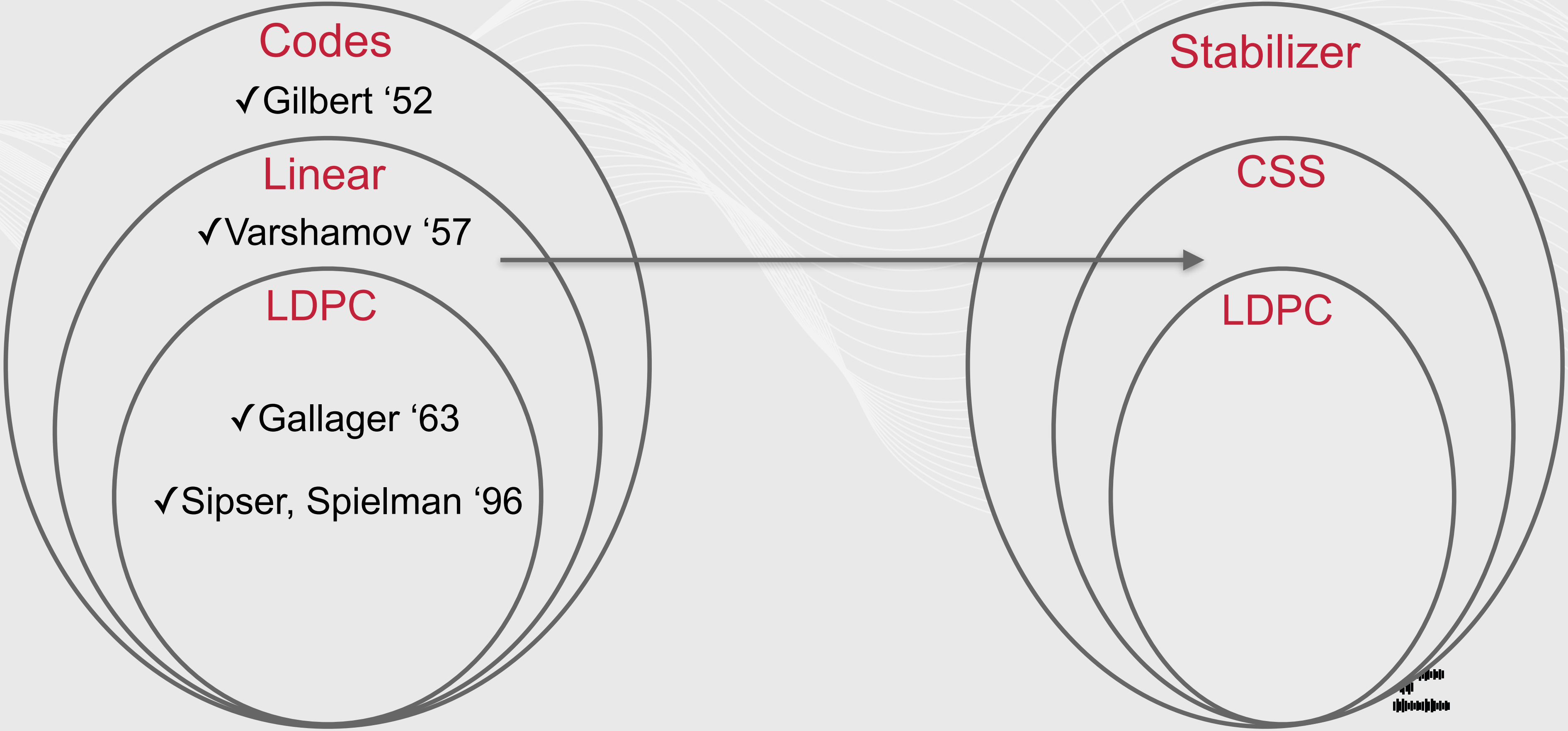

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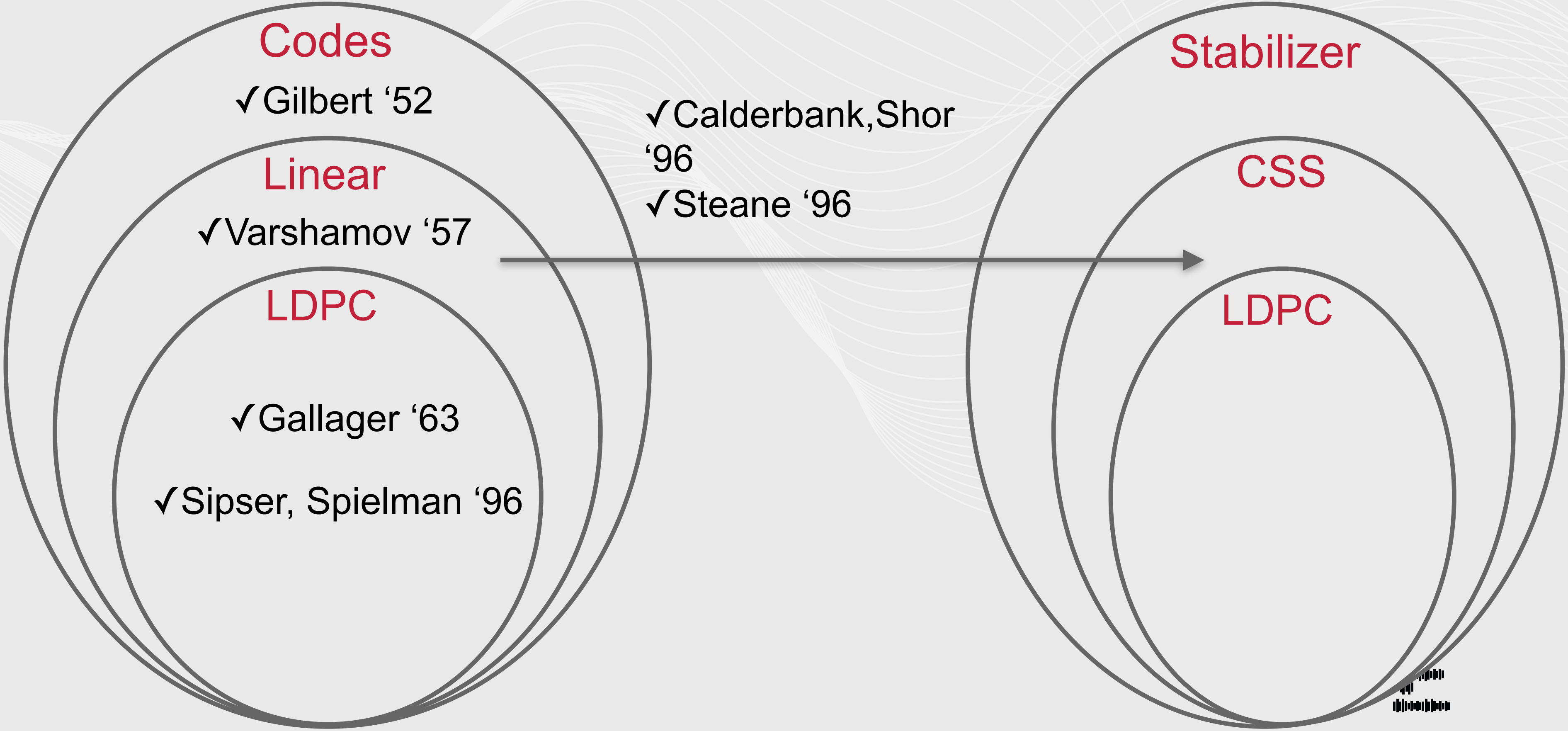

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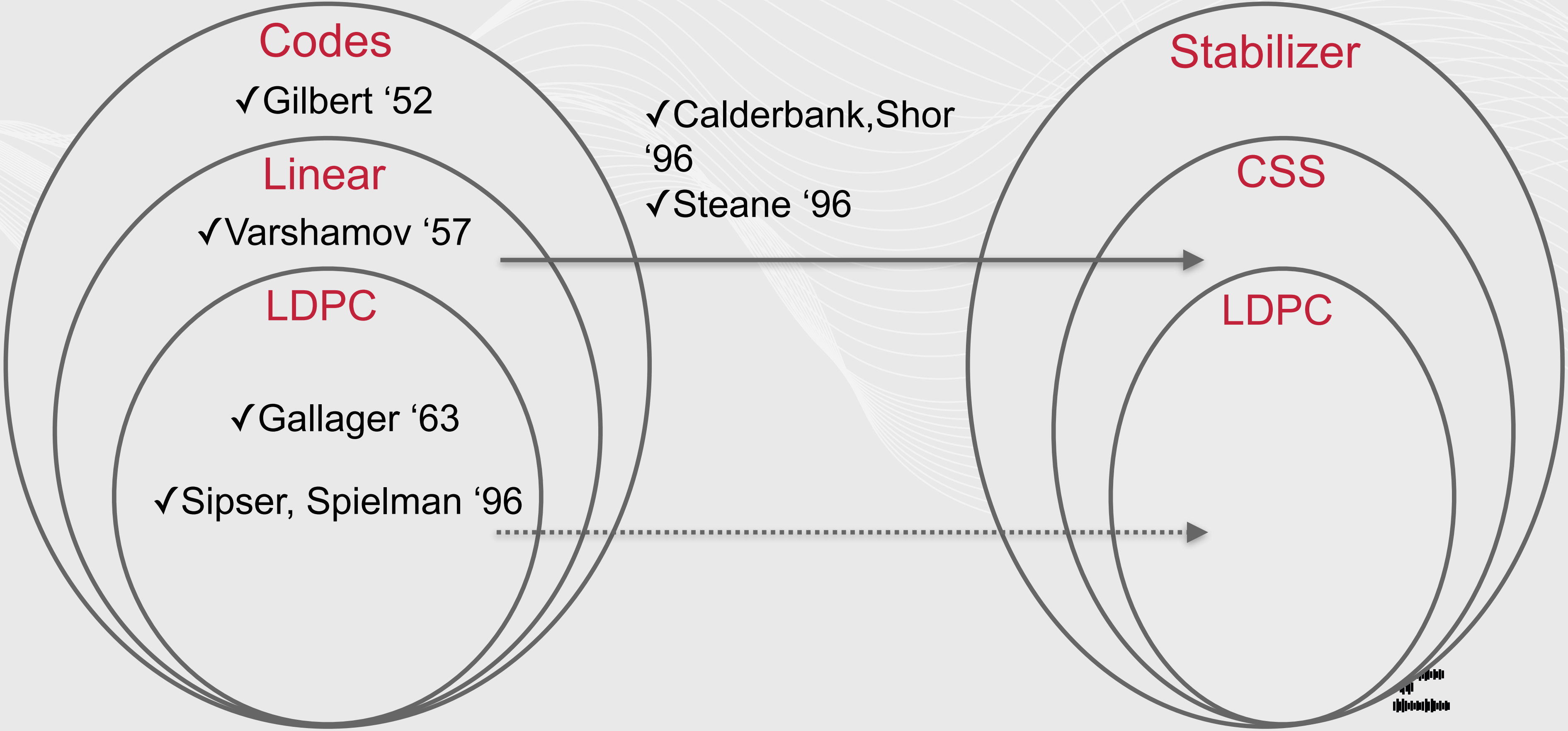

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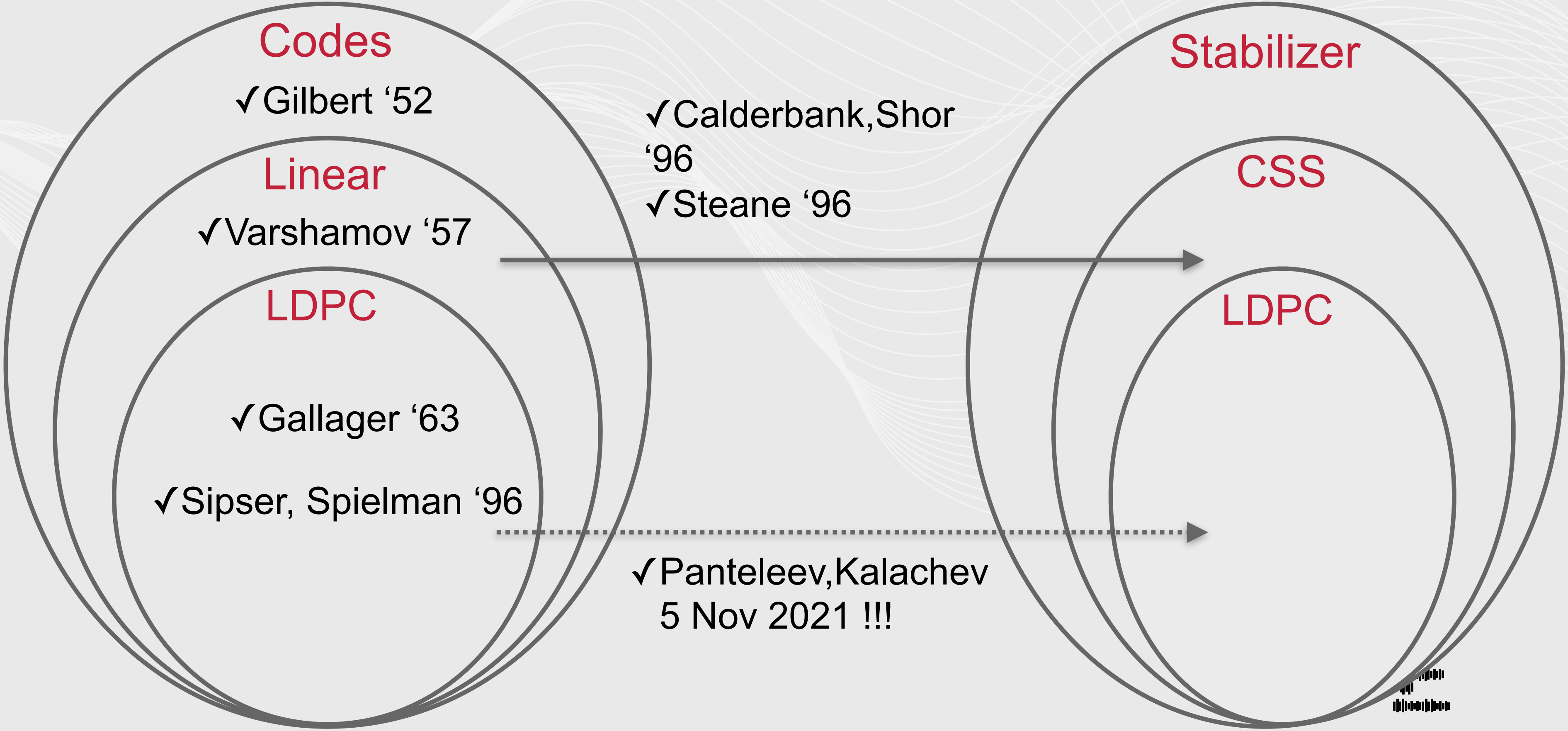

Goal - To build good quantum codes

1a.

The Classical World

Codes from expanding graphs

Graphs are easier to build

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- Linear Codes -

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- Expansion trick - Sipser-Spielman'96 - If G is an expander, then the resulting code is good !!

2.

The Quantum World

Chain Complex Perspective

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  - $d = \min(d(\mathscr{C}), d(\mathscr{C}^*))$

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- Key technique - Tensor product

Improvements in distance

Up to $\sqrt{N} \log^{1/4} N$

Topology

Use cohomology theories on manifolds

$\sqrt{N}$ to $\sqrt{N} \log^{k} N$

Expansion

Take tensor products of expanders

$N^{0.6}$ to N

Symmetry

Use group symmetry to quotient

2a.

Tensor Product

Definition
Key Lemma

Tensor Product

[redacted]

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- Given complexes, $\mathscr{C}, \mathscr{D}$, the tensor product is $(\mathscr{C} \otimes \mathscr{D})_k = \bigoplus_{i=1}^{k} C_i \otimes D_{k-i}$.

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  - $\mathscr{C} \otimes \mathscr{D} = C_1 D_1 \to C_1 D_0 \oplus C_0 D_1 \to C_0 D_0$

Distance lemma

[redacted]

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  - Kaufman, Tessler ’20

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TZ'15,EKZ'20, KT'20

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2b.

Symmetry

If you can’t increase d, decrease n

How can symmetry help?

- Let $\mathscr{C} = \operatorname{span}\big((1,0,1,0), (0,1,0,1)\big) \subseteq V = \mathbb{F}_2^4$. It has relative distance $2/4 = 1/2$.

- Let $H = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $(\sigma_1, \sigma_2) \in H$.
  - H acts on V such that σ_1 permutes first two coordinates and σ_2 permutes the other two.

- Quotient is the image of the projection $\varphi : V \to V/H$ where $\varphi(a, b, c, d) = (a + b, c + d)$.
  - $\varphi(\mathscr{C}) = \mathscr{C}/H = \operatorname{span}((1,1))$ has relative distance 1.

Results

[redacted]

Results

- Hastings, Haah, O'Donnell '20 -
- Panteleev, Kalachev '20 -
- Breuckmann, Eberhardt '20 -
- Panteleev, Kalachev '21 -

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HHO’20,PK’20, BE’20, PK’21

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- Quantum Locally Testable Codes - Applications to NLTS, qPCP conjecture.

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