Explicit Abelian Lifts and Quantum LDPC Codes

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- Main goal is to explicitly build symmetric expanding graphs

- Let us see why and how!

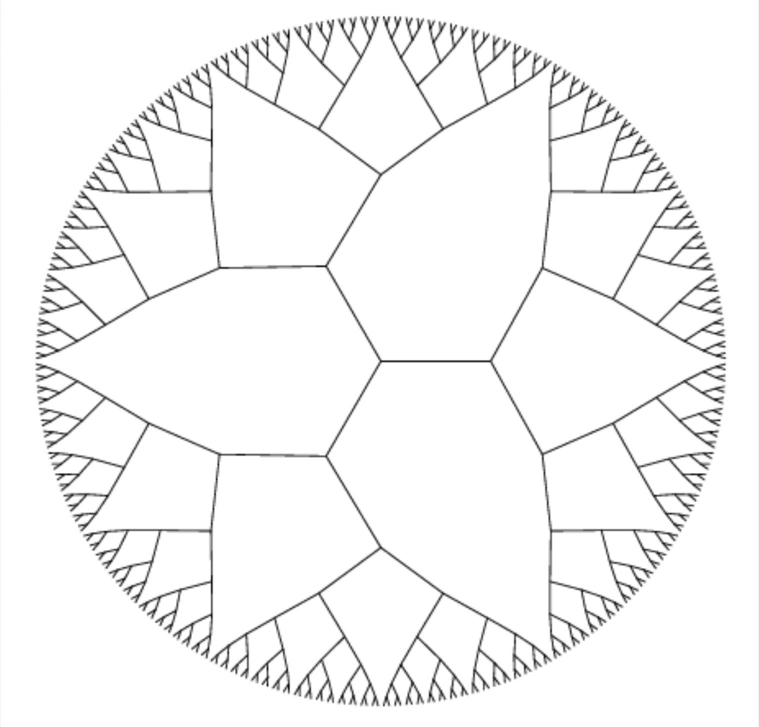

Image credits - Hoory, Linial, Wigderson ‘06

OUTLINE

1. **Introduction**

   Motivation and history

2. **Our Results**

   Statement and an application

3. **Key Contribution**

   A better count via DFS!

4. **Conclusion**

   Summary and open Problems

1.

Introduction

Here we go!

Expansion

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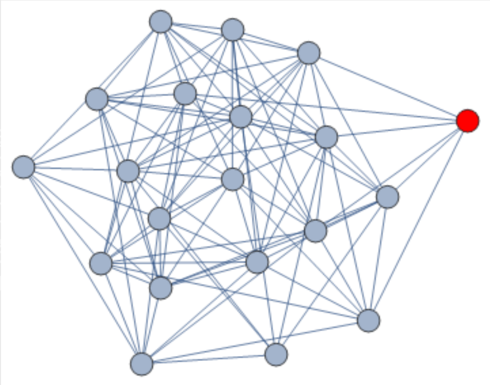

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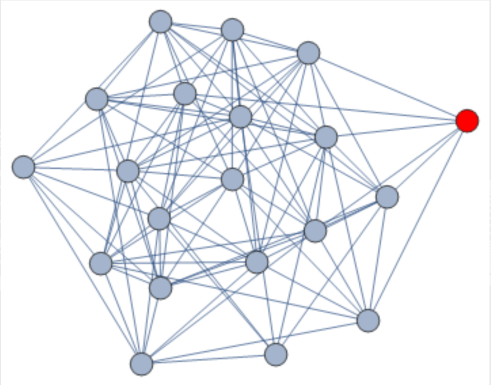

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- Q - Given d, ε, can we construct infinite families of d-regular graphs $\{G_n\}$ with $n \to \infty$ such that $\lambda(G_n) \leq \varepsilon d$?
  - Alon-Boppanna bound says that the best possible is $2\sqrt{d-1} - o_n(1)$.

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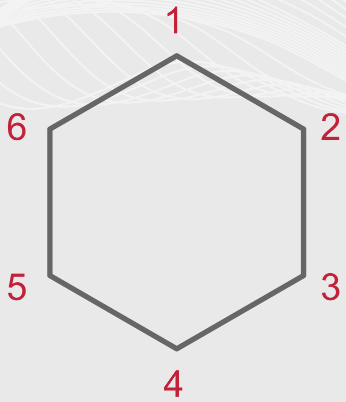

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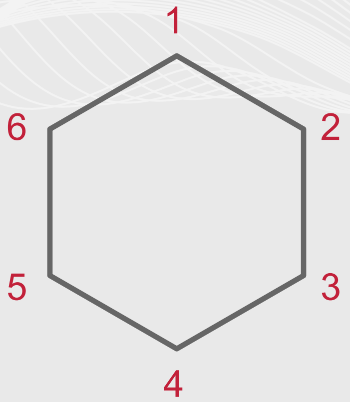
- Eg - If $H = \mathbb{Z}_6$, we have the cycle graph C_6 such that
  - $\mathbb{Z}_6 \subseteq \mathrm{Aut}(C_6)$ as $i : n \to n + i \mod 6$.



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Q - Can we have both ?

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- Property Testing - Interesting work by [Goldreich-Wigderson'21] builds expander graphs with $\mathrm{Aut}(X) = \{\mathrm{id}\}$ and shows applications to property testing.

Q - For a given family of groups H_n, can we explicitly construct a family of expander graphs G_n such that $H_n \subseteq \mathrm{Aut}(G_n)$?

Known Techniques

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- Algebraic Constructions - Specific constructions for certain groups like $\mathrm{PSL}_2(\mathbb{F}_q)$ but are highly non-elementary.

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- Group-based lifts (Covering maps) - A generic technique introduced by Bilu, Linial'06 in context of graphs. A special case of the topological notion of covering maps.
  - Used extensively to construct expanders.

(H, ℓ) lift of a graph

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$$\begin{tikzcd} G(s) \\ G \arrow[u] \end{tikzcd}$$

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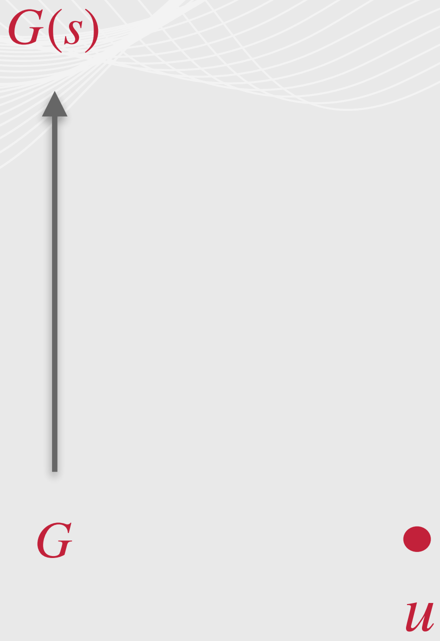

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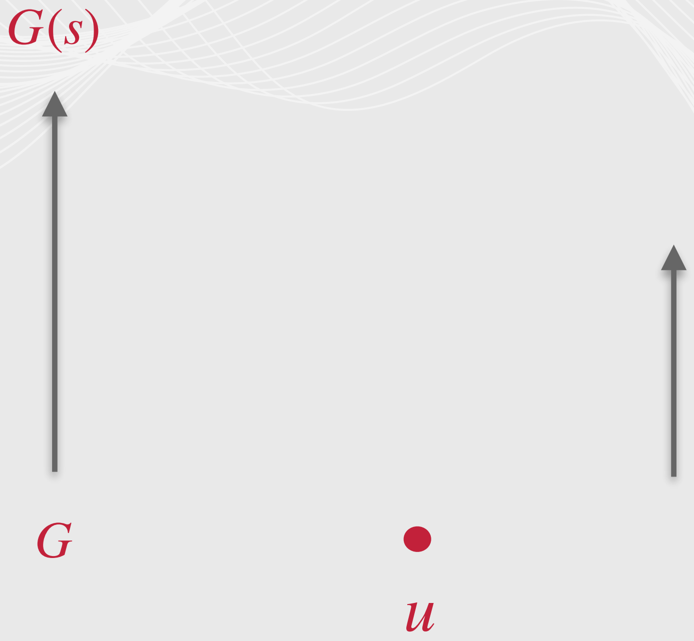

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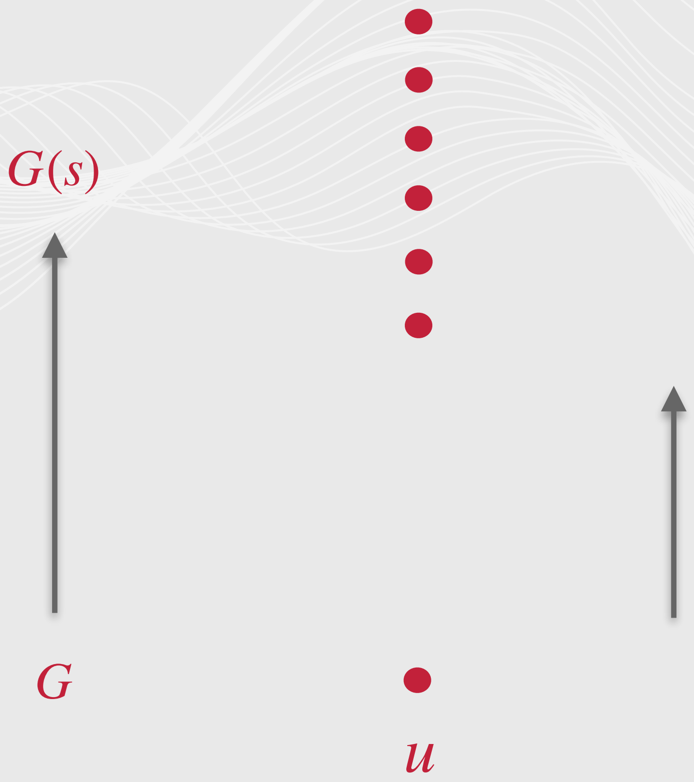

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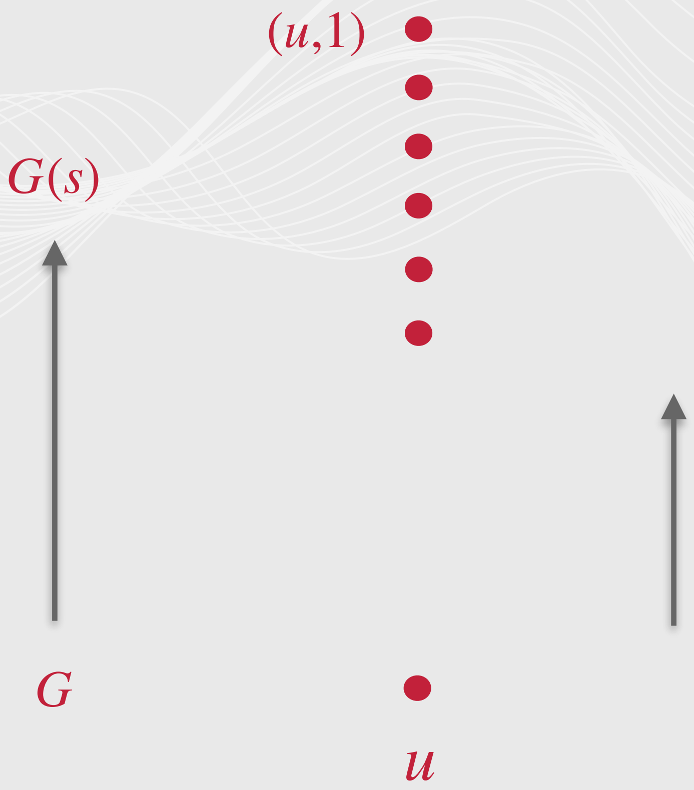

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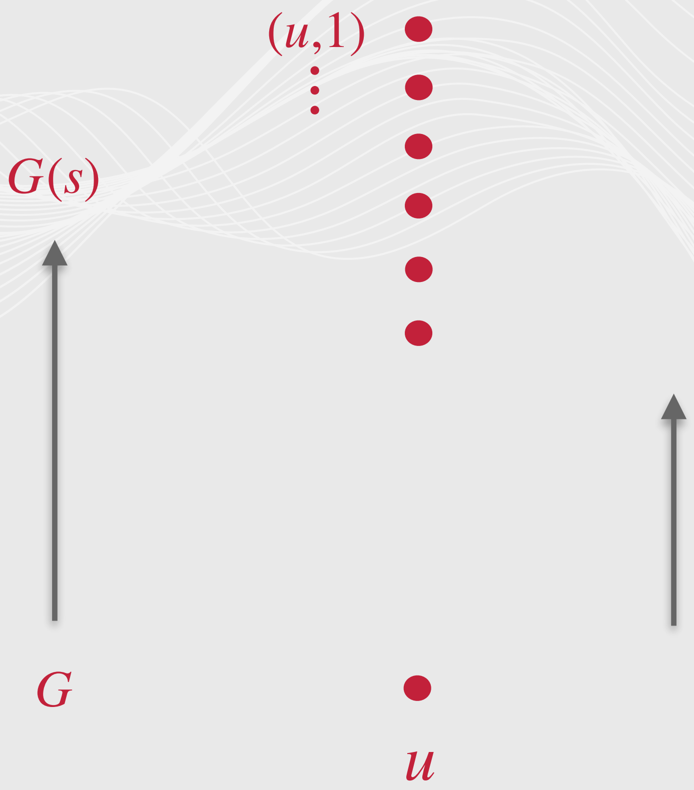

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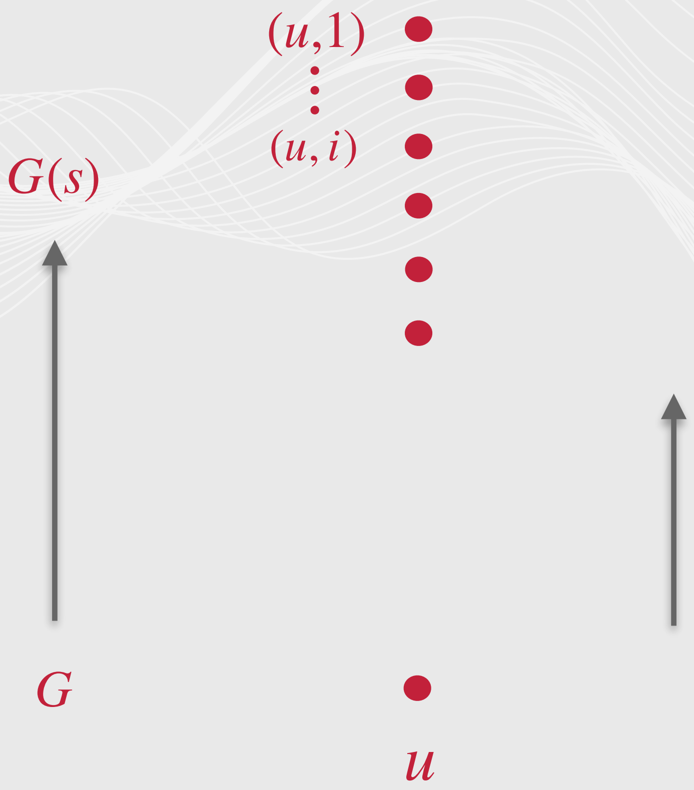

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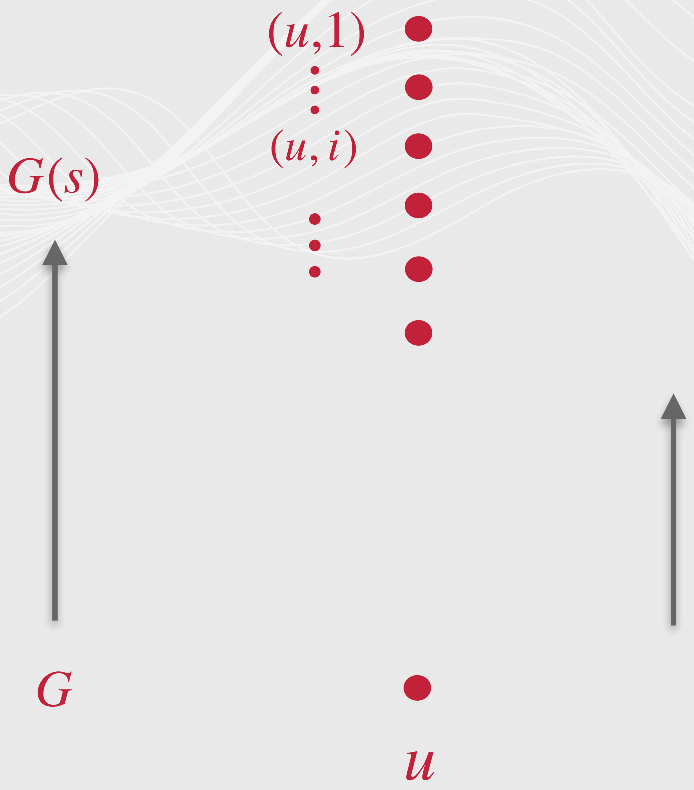

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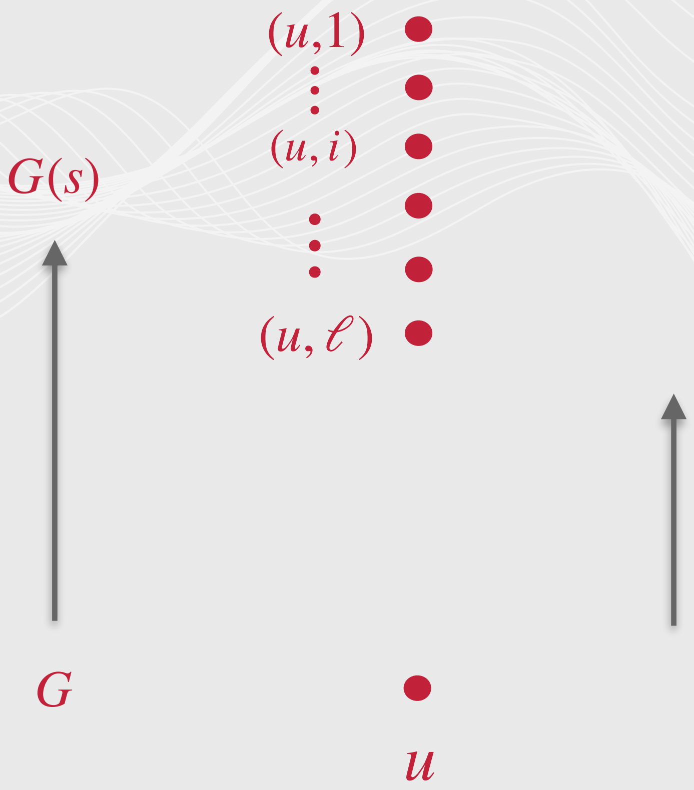

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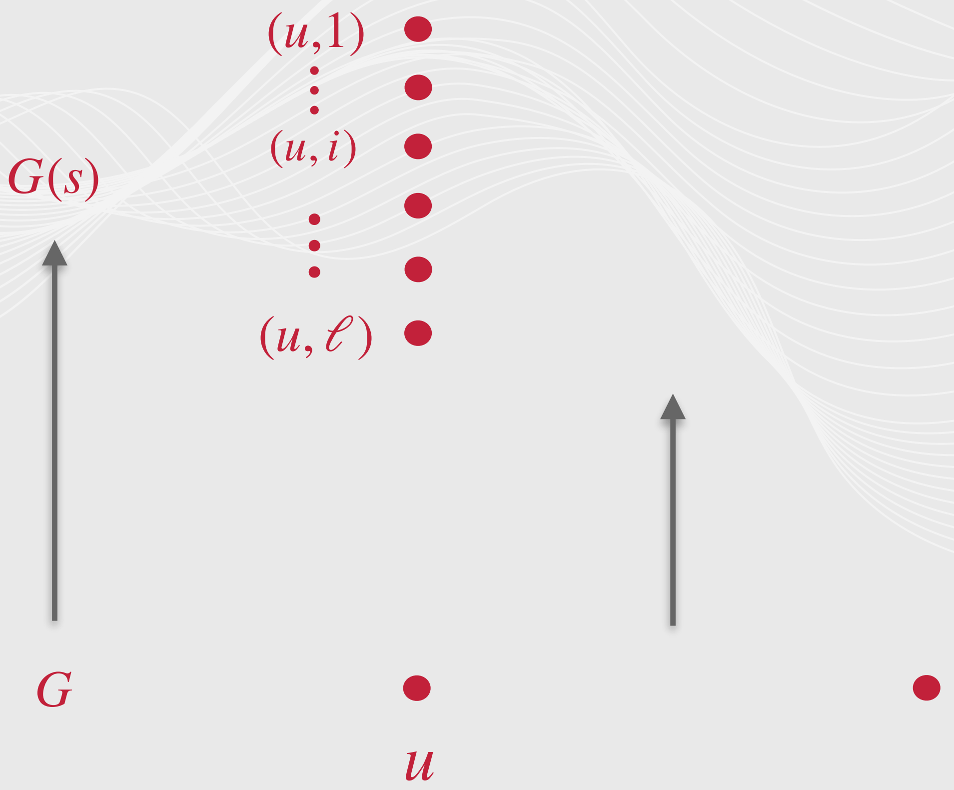

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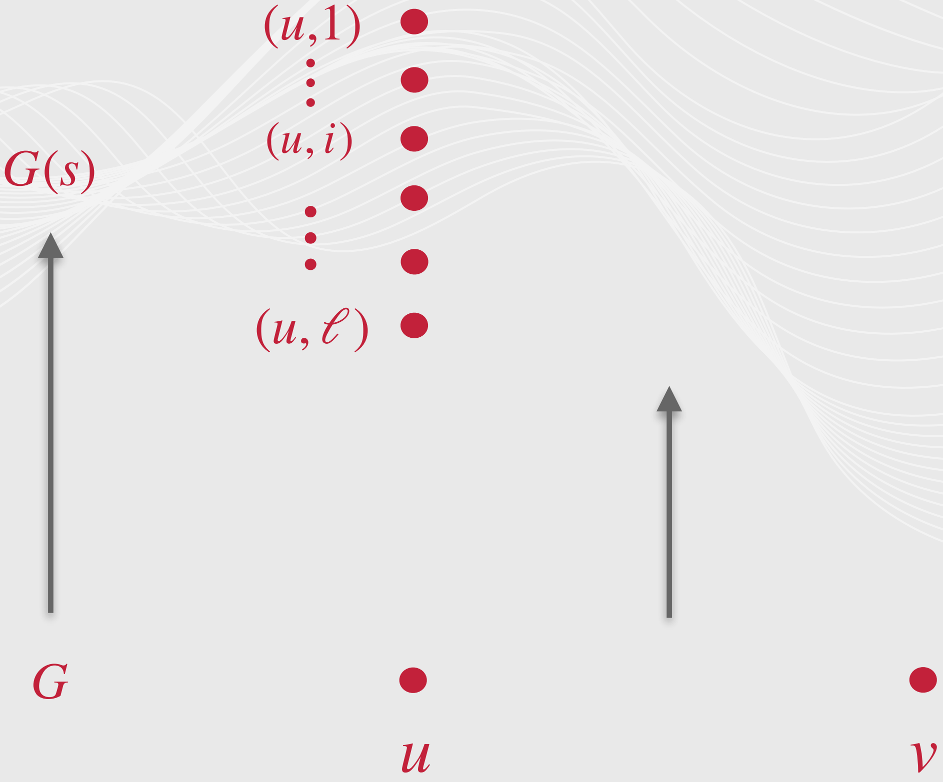

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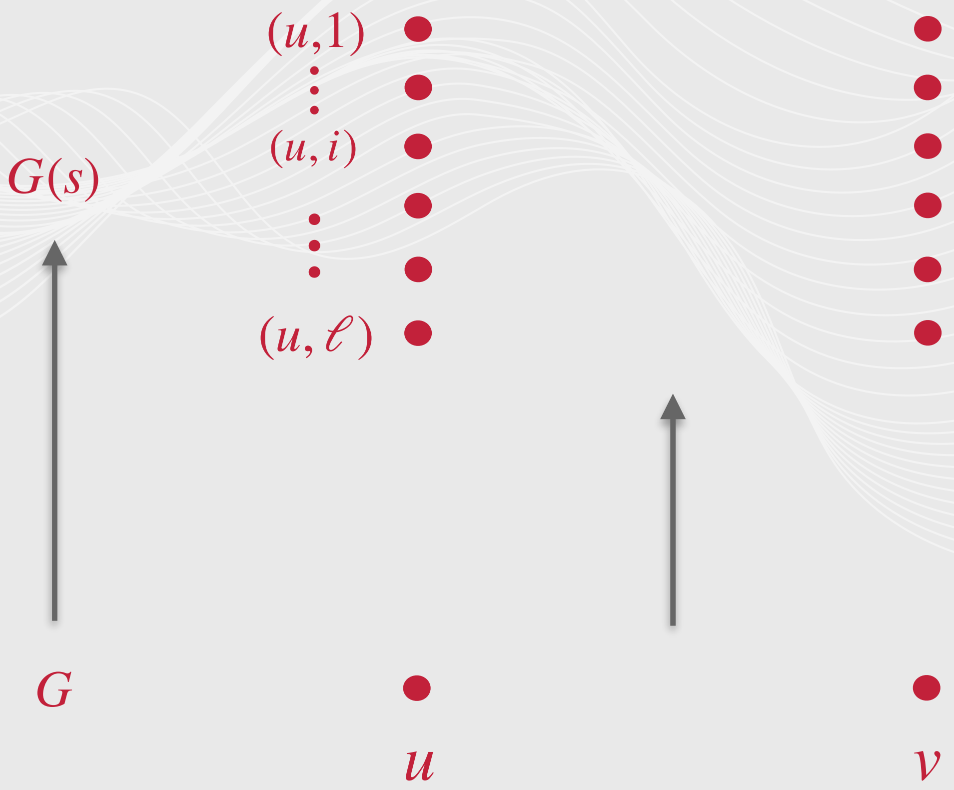

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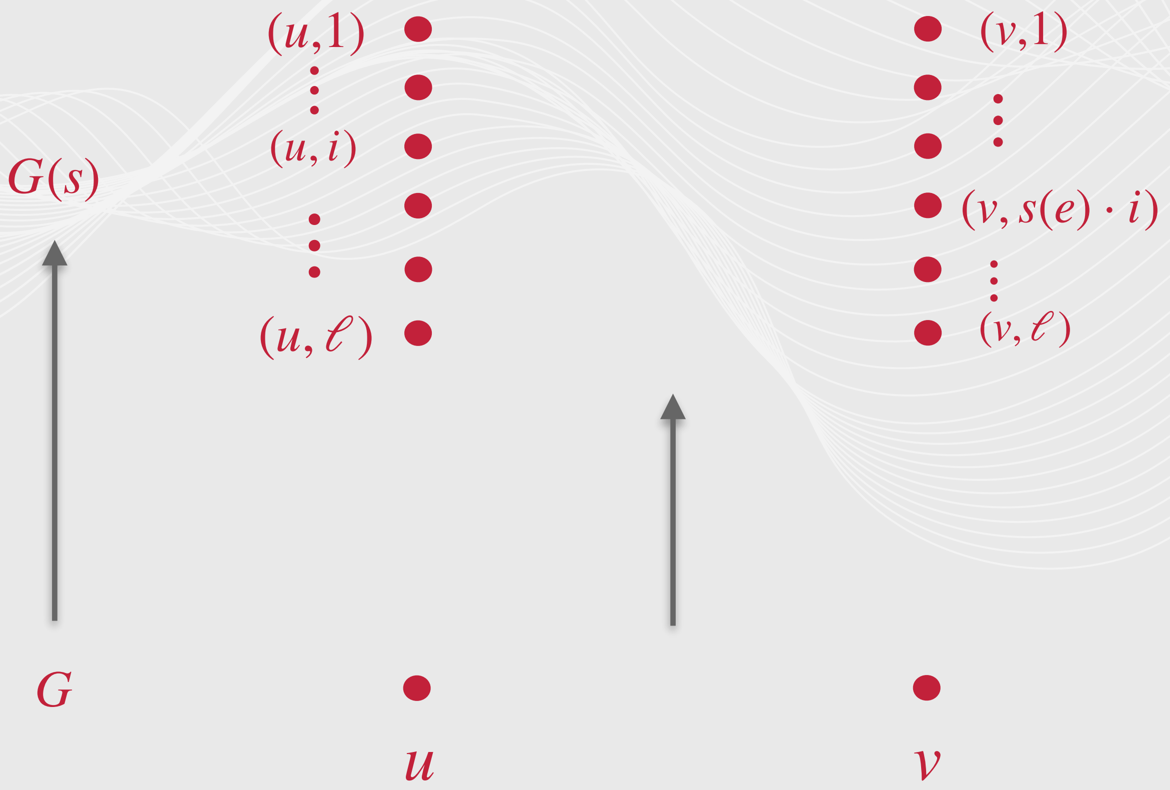

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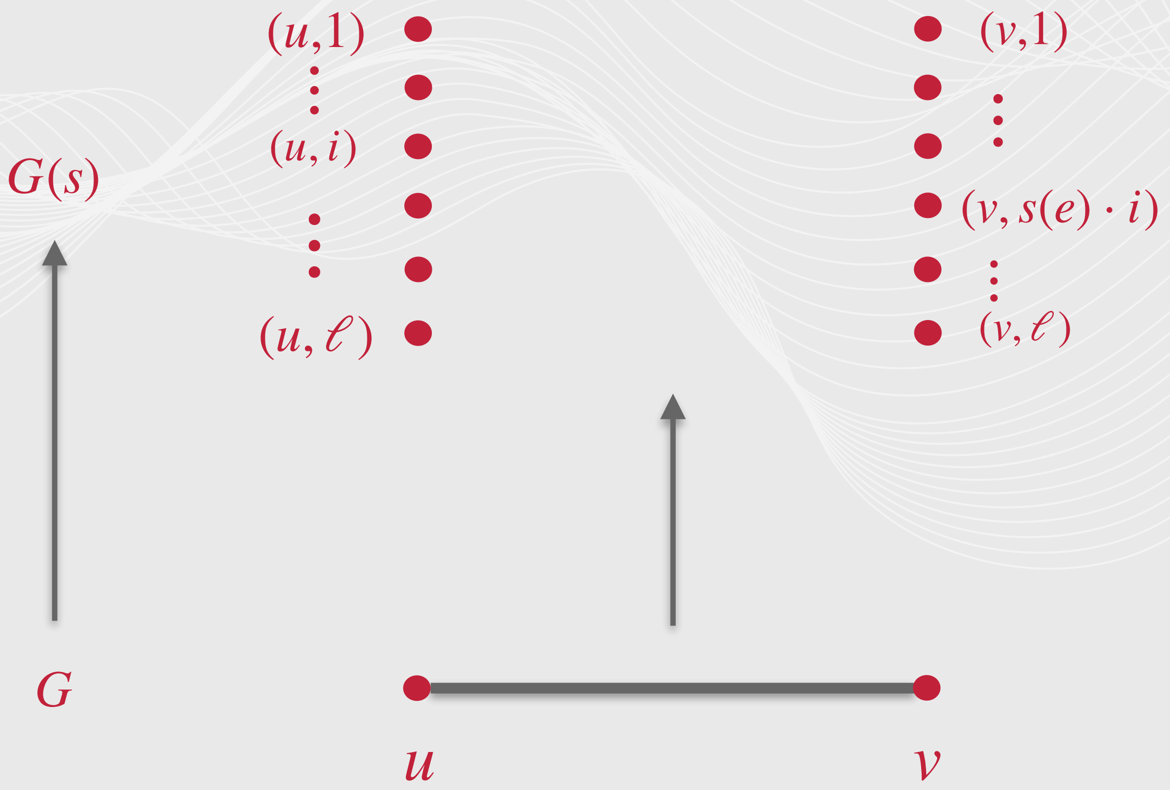

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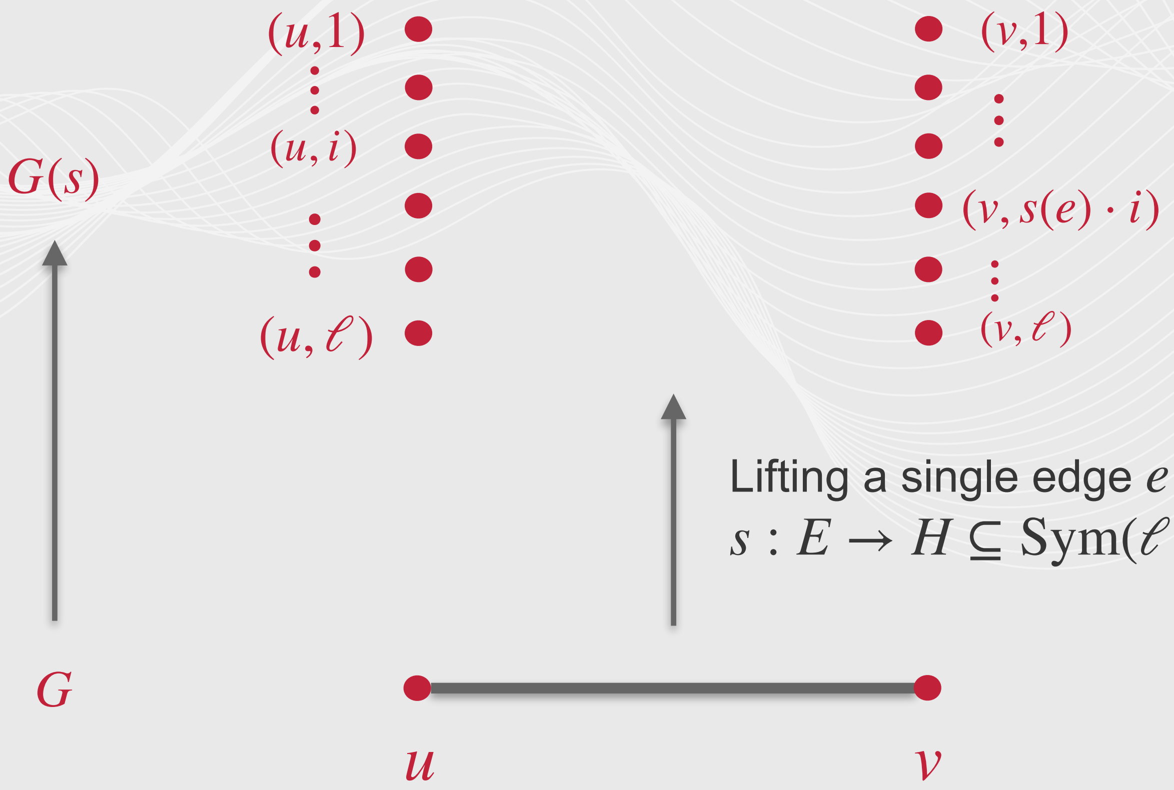

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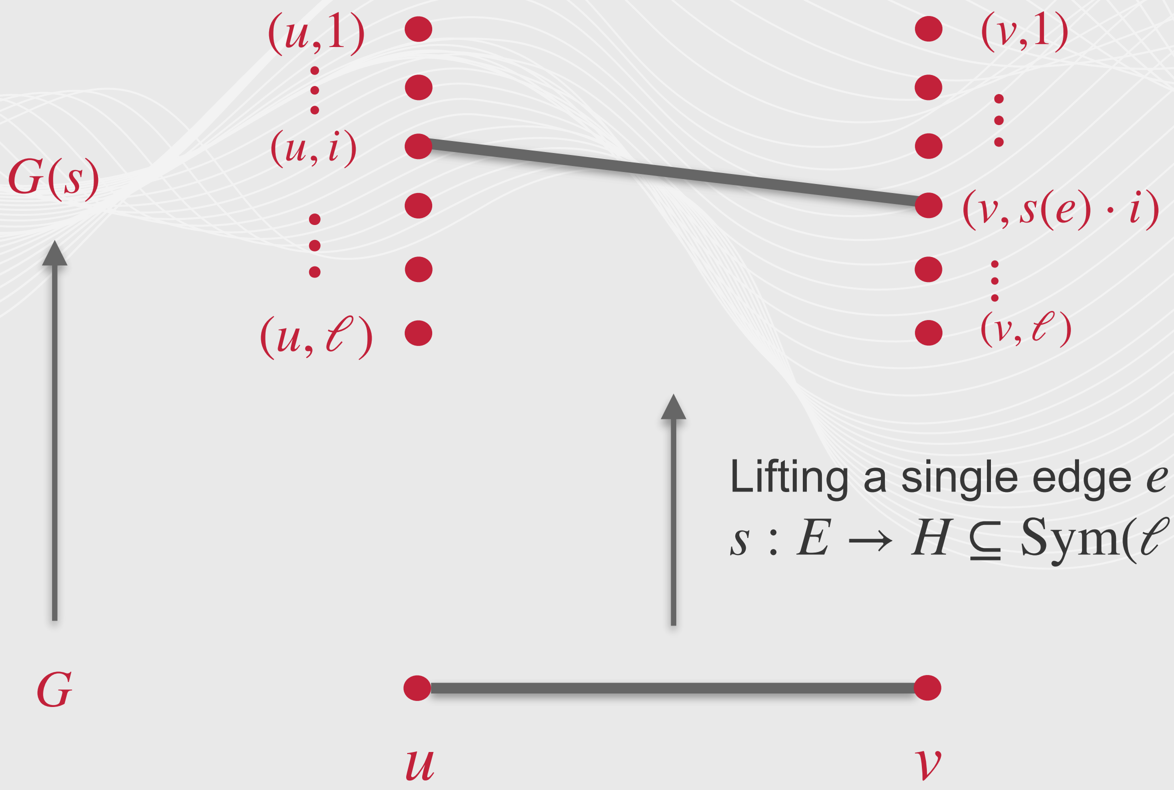

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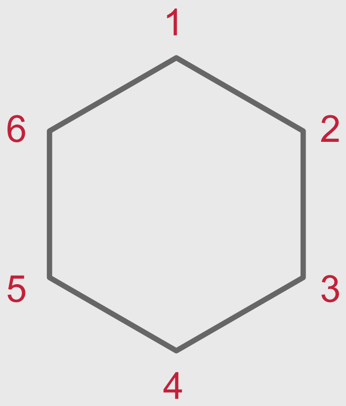

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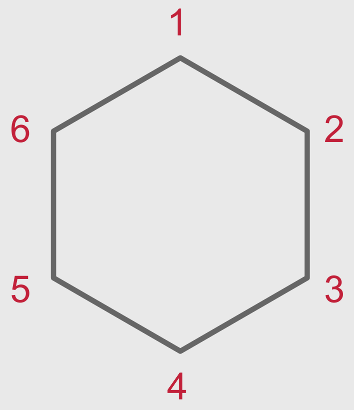

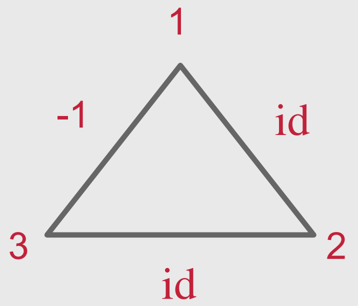

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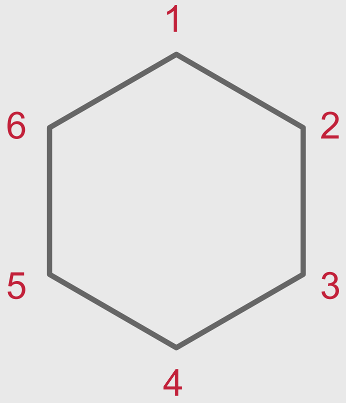

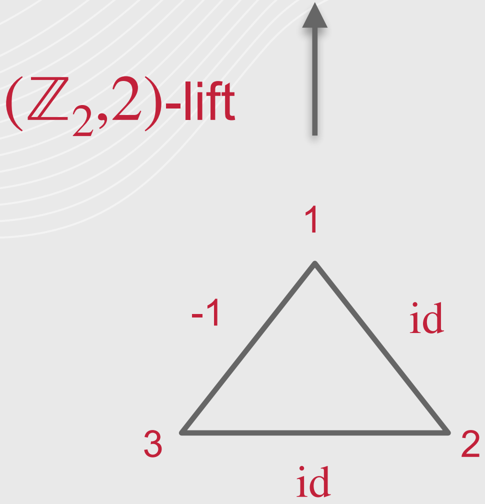

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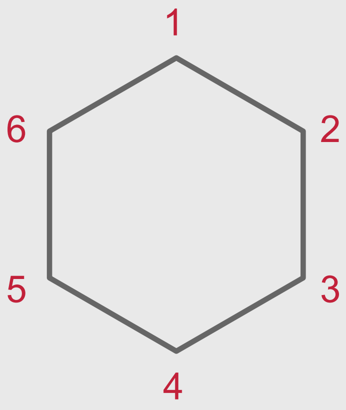
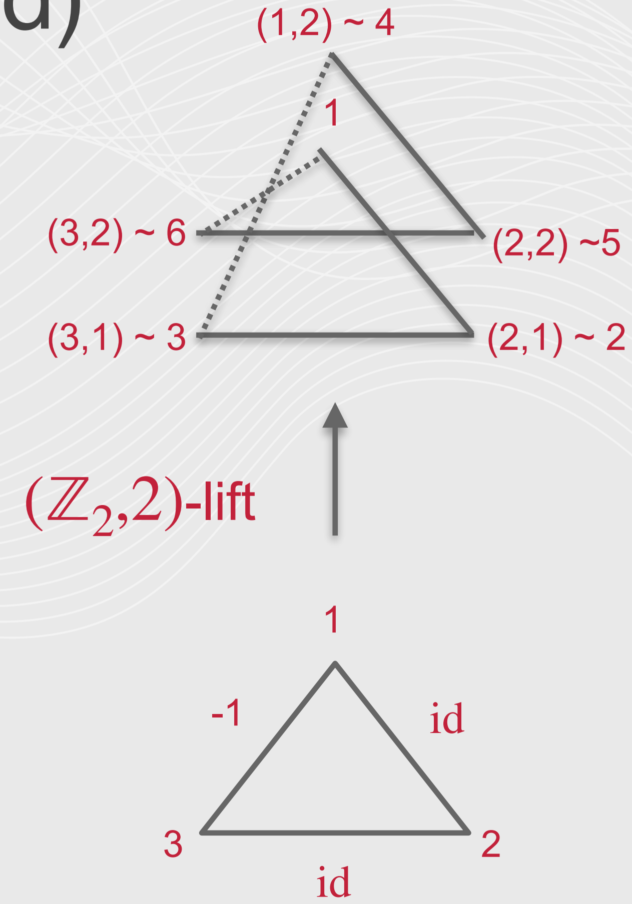




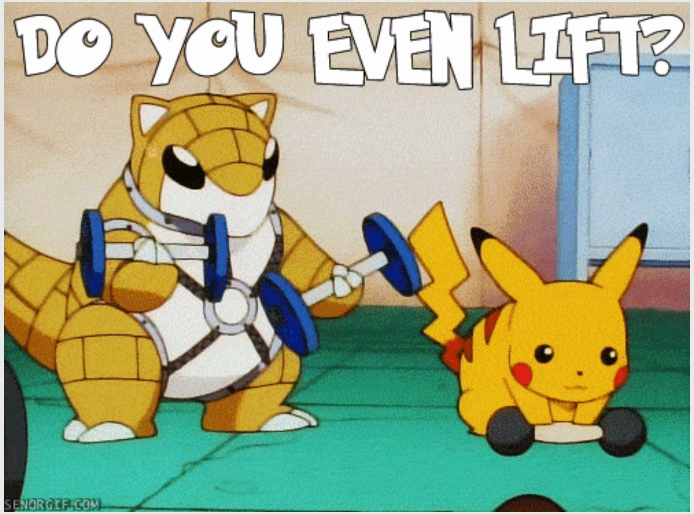
Properties of lifting

- Explicit characterization of the spectrum of lifted graph, $G(s)$.
- Preserves degree.
- If H is abelian, it possesses symmetries of H i.e.,
  - $H \subseteq \mathrm{Aut}(G(s))$.
- If G is an expander and s is random, $G(s)$ is known to be an expander*. Challenge is to explicitly construct such a signing s.

Quick history of lifting

Technique	Authors	Lift	$\lambda(G)$	Explicit
Discrepancy	[Bilu, Linial '06]	2-lift	$\sqrt{d}\log^{1.5} d$	Yes
	[Agrawal, Chandrashekharan, Kolla, Madan '16]	$(\mathbb{Z}_\ell, \ell)$	$O(\sqrt{d})$	No
Method of interlacing polynomials	[Marcus, Spielman, Srivastava '13] [Cohen '16]	2-lift	$2\sqrt{d-1}$	Yes
	[Hall, Puder, Sawin '15]	(H, ℓ) for some non-abelian		No?
Trace Power Method	[Mohanty, O'Donnell, Paredes '20]	2-lift	$2\sqrt{d-1}+\varepsilon$	Yes

Can we lift more?



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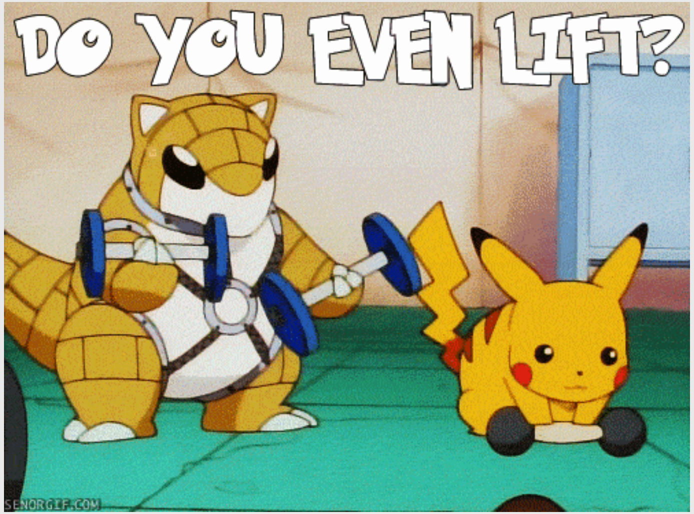
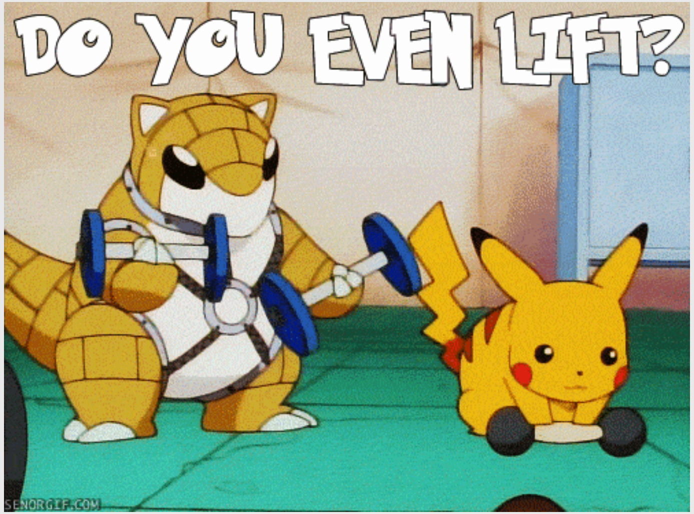


Can we lift more?

- [ACKM'16] showed that for $\mathbb{Z}_\ell$, there exists good signings for $\ell \le 2^{n/d^3}$.
  - They further show that for any abelian group H, no lift of size $\ell > \exp(nd)$ is expanding.

- The goal now is to construct $(\mathbb{Z}_\ell, \ell)$ lifts for $3 \le \ell \le 2^{nd}$.



2.

Our Results

Yes, we do lift!
And that too explicitly!

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Theorem - For any $d \geq 3$, large enough n and "nice" $\ell(n)$, we have an explicit family of d-regular expanding graphs $\{G_n\}$ such that G_n is a $(\mathbb{Z}_{\ell(n)}, \ell(n))$-lift* of some base graph.

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Main Result

Technique	Authors	Lift	$\lambda(G)$	Explicit
Discrepancy	[BL06]	$(\mathbb{Z}_2, 2)$	$\tilde{O}\left(\sqrt{d}\right)$	Yes
	[ACKM16]	$(\mathbb{Z}_\ell, \ell)$ $\ell \le \exp(n/d^3)$	$O\left(\sqrt{d}\right)$	No
	This work	$(\mathbb{Z}_\ell, \ell)$ $\ell = \exp(\Theta(n))$	$\tilde{O}\left(\sqrt{d}\right)$	Yes
Trace Power Method	[MOP20]	$(\mathbb{Z}_2, 2)$	$2\sqrt{d-1} + \varepsilon$	Yes
	This work	$(\mathbb{Z}_\ell, \ell)$ $\ell \le \exp(n^{\delta(d,\varepsilon)})$	$2\sqrt{d-1} + \varepsilon$	Yes
		$(\mathbb{Z}_\ell, \ell)$ $\ell \le \exp(n^{0.01})$	εd	

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3.

Key Contribution

A better count of non-backtracking hikes

Trace Power method

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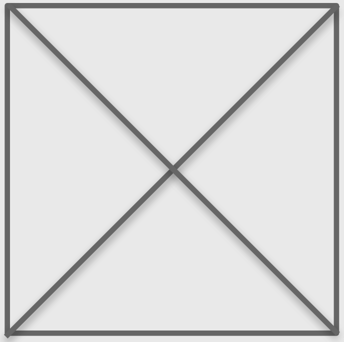

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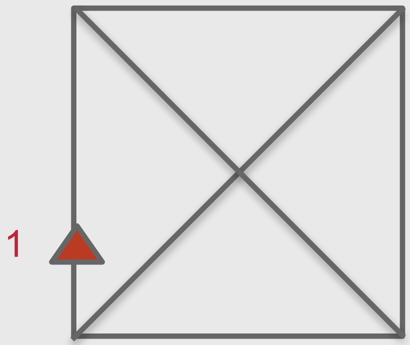

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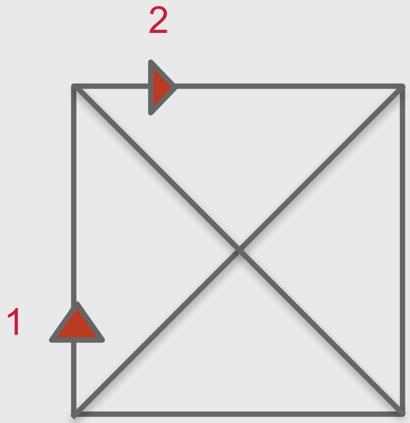

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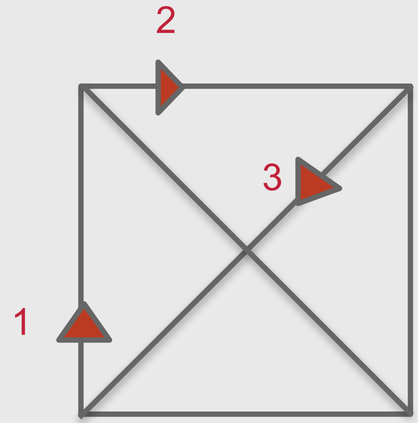

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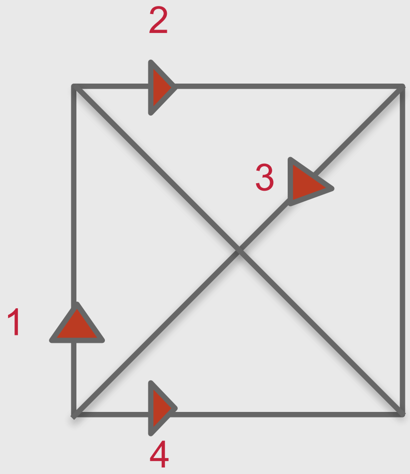

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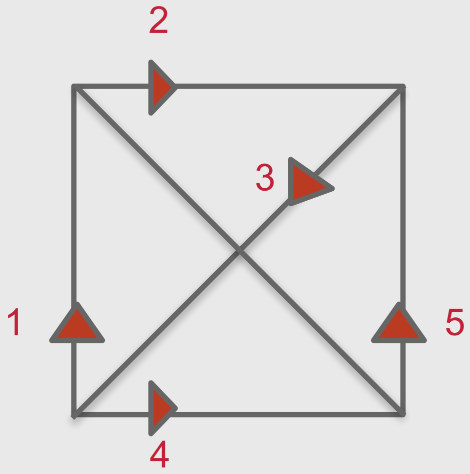

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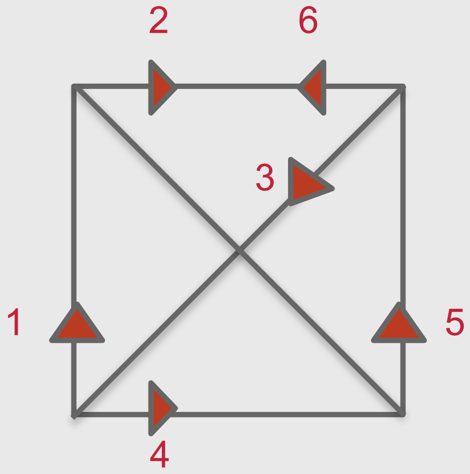

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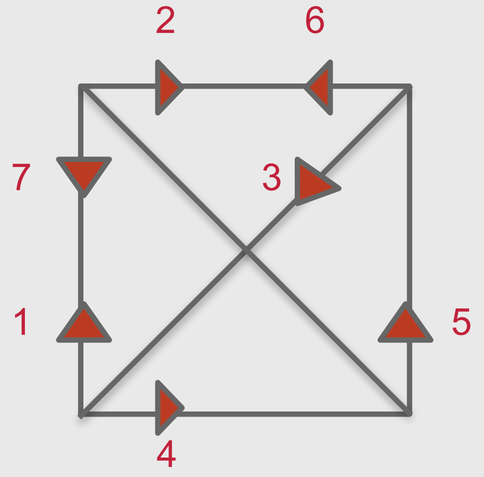

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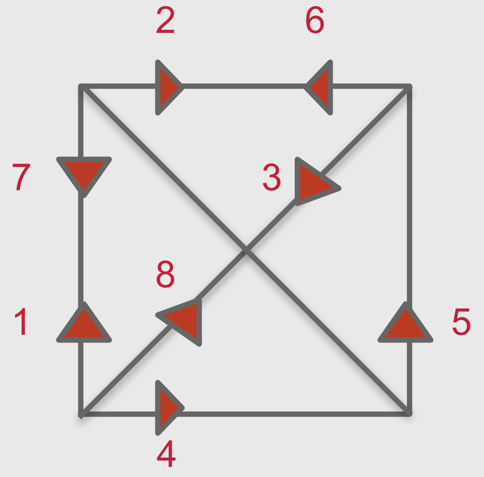

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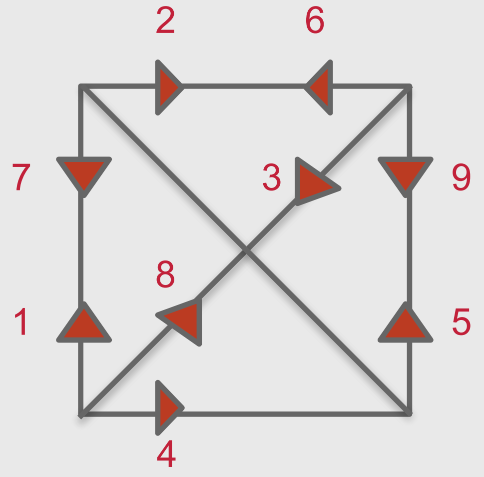

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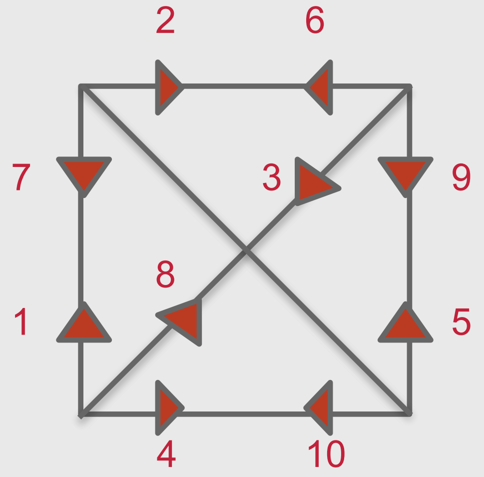

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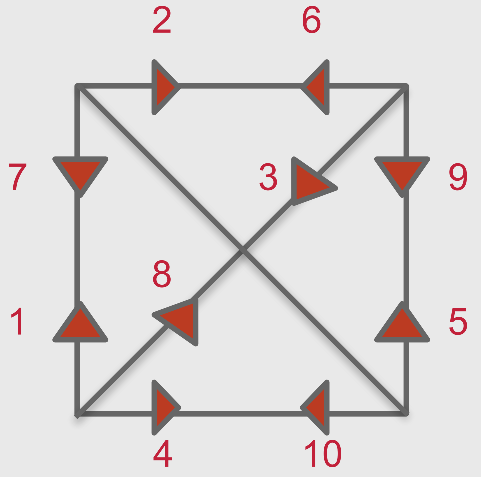

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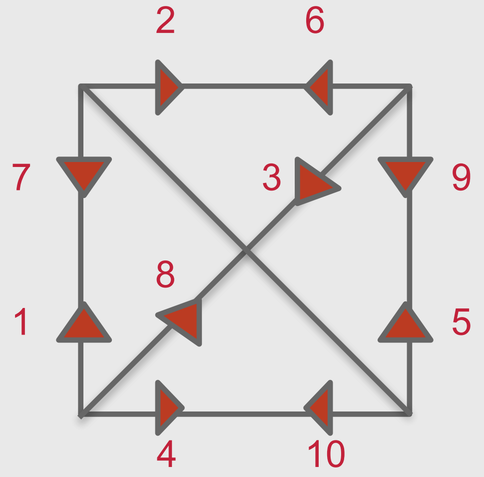

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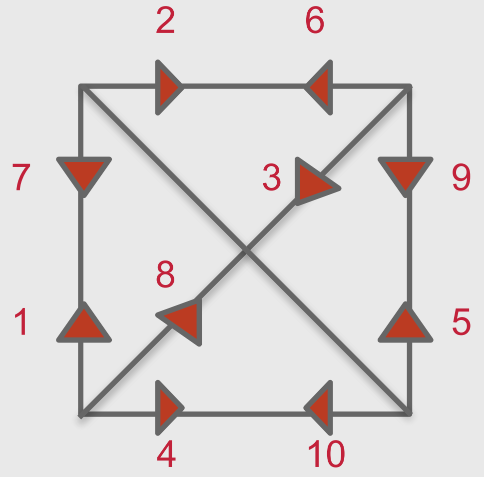
- Ideal Count - $(d-1)^k$ would give the optimal bound of $2\sqrt{d-1}$.



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- We extend the near-optimal bound to walks of length $O(n^{\delta(d,\varepsilon)})$ and obtain a weaker bound all the way up to $k = O(n^{0.01})$.

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- Then, count the number of hikes corresponding to a given graph.

4.

Conclusion

All good things come to an end

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- As an application, we get new explicit LDPC codes — classical and quantum.

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- Generalize the result to new families of non-abelian groups.

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