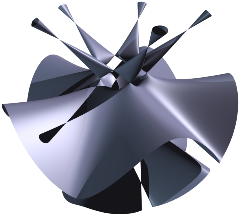

Gröbner basis - What, Why and How?

Tushant Mittal

Agenda

Motivational Problems

- **Ideal Membership Problem**
  Given $f \in k[x_1, x_2, \cdots x_n]$ and an ideal $I =< f_1, f_2, \cdots, f_n >$, determine if $f \in I$.

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  Find all solution in k^n of a system of polynomial equations $f_i(x_1, x_2, \cdots , x_n) = 0$. In other words, given an ideal I, compute $V(I)$.

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  Given a parametric solution of x_i's in terms of variables t_i i.e. $x_i = g_i(t_1, t_2, \cdots, t_i)$, find a set of polynomials f_i such that $x_i \in V(< f_1, f_2, \cdots, f_n >)$. It can be easily observed that this is essentially the inverse of the above question i.e given $V(I)$ compute I.

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But an immediate question arises.
How do we even store these ideals which are possibly of infinite size ?

Noetherian Ring

- A Noetherian ring is a ring that satisfies the ascending chain condition on ideals; that is, given any chain of ideals:

$$I_1 \subseteq \cdots \subseteq I_{k-1} \subseteq I_k \subseteq I_{k+1} \subseteq \cdots$$

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Theorem (Hilbert Basis Theorem)

R is Noetherian $\Rightarrow R[x]$ *is Noetherian*

Special Cases

- $R = k[x]$ i.e. $n = 1$.
  We know that $k[x]$ is a PID. Moreover, it is a Euclidean domain and hence, a polynomial $g \in < f >$ *iff* $f|g$.

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- Linear Algebra techniques can be used efficiently when the degree of the polynomials is restricted to 1 irrespective of n.
- We will generalize both the idea of division and a basis to solve the problem for the general case.

Monomial Ordering

We will use the notation x^α to represent $\prod_i^n x_i^{\alpha_i}$ where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$.

Definition (admissible ordering of monomials)

A total ordering on all monomials is an ordering for which holds:

- $x^\alpha < x^\beta \Rightarrow \forall\delta\colon\ x^\alpha x^\delta < x^\beta x^\delta$.
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A few popular orderings are:

1. Lexicographical ordering: In which we compare x^α and x^β thus: if the first $k-1$ indices agree, $\alpha_i = \beta_i, i \leq k-1$ and the kth differ, we decide based on that index $\alpha_k \leq \beta_k \Rightarrow \alpha \leq \beta$, and the reverse.

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2. Graded lexicographical order: in which the order is by the degree of the monomials and ties are broken using lexicographical ordering.

Preliminary Definitions

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Definition (Leading Monomial)

$LM(f) = x^{multideg(f)}$

Definition (Leading Term)

$LT(f) = LC(f)LT(f)$

Example

Let $f = 7x^3y^2z + 2x^2yz^4 + 9xy^4 + 3yz^7 + 2$.

Using the lex ordering,

- $multideg(f) = (3, 2, 1)$
- $LC(f) = 7$
- $LM(f) = x^3y^2z$
- $LT(f) = 7x^3y^2z$

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Whereas using the grlex ordering we would get,

- $multideg(f) = (0, 0, 7)$
- $LC(f) = 3$
- $LM(f) = yz^7$
- $LT(f) = 3yz^7$

Division Algorithm

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 & \overline{\qquad\qquad\quad 0} & \to x + y + 1
\end{array}
$$

Division Algorithm

Algorithm 1: Multi_Divide$(f, f_1, f_2, \cdots f_n)$

$a_1 := 0; a_2 := 0; \cdots a_n := 0; r = 0$
$p := f$
while $p \neq 0$ **do**
    $i := 1$
    divisionoccured := false
    while $i \leq s$ *AND divisionoccured* := *false* **do**
        if $LT(f_i) | p$ **then**
            $a_i := a_i + LT(p)/LT(f_i)$
            $p := p - (LT(p)/LT(f_i))f_i$
            divisionoccured := true
        else
            $i := i + 1$
    if *divisionoccured* := *false* **then**
        $r := r + LT(p)$
        $p := p - LT(p)$
return $a_1, a_2, \cdots, a_n, r$;

Are we done? NO!!

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$$Multi\_Divide(xy^2 - x, xy + 1, y^2 - 1) = (y, 0, -(x + y))$$
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- We want to find a "good" basis for a given ideal which preserves the property that nonzero remainder implies non-membership also called the *remainder property*

Does such a basis exist ? Is it computable ?

Gröbner basis

Definition

Fix a monomial order. A finite subset $G = \{g_1, g_2, \cdots, g_n\}$ of an ideal I is said to be a Gröbner basis (or standard basis) if

$$< LT(g_1), LT(g_2) \cdots , LT(g_n) > = < LT(I) >$$

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Theorem

Let G be a Gröbner basis for an ideal I and let $f \in k[x_1, \cdots, x_n]$. Then there is a unique remainder r on division by G with the following two properties:

1. *No term of r is divisible by any of $LT(g_1), \cdots LT(g_n)$.*
2. *There is $g \in I$ such that $f = g + r$.*

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$$S(f,g) = \frac{x^\gamma}{LT(f)} f - \frac{x^\gamma}{LT(g)} g$$

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Lemma

Suppose we have a sum $\sum_{i=1}^{n} c_i f_i$, where $c_i \in k$ and $multideg(f_i) = \alpha$. If $multideg(\sum_{i=1}^{n} c_i f_i) < \alpha$, then

$$\sum_{i=1}^{n} c_i f_i = \sum_{i=1}^{n} c'_{ij} S(f_i, f_j)$$

Buchberger's Criterion

Theorem (Buchberger '65)

Let I be a polynomial ideal. Then a basis $G = g_1, \cdots g_n$ for I is a Gröebner basis for I if and only if for all pairs $i \neq j$, the remainder on division of $S(g_i, g_j)$ by G is zero.

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Algorithm 3: Buchberger(F)

```
Start with G:= F
do
    G' := G
    for pair of polynomials f1, f2 ∈ G' do
        h := remainder[G, S(f1, f2)]
        if h ≠ 0 then
            G = G ∪ {h}
while G ≠ G';
output G
```

Using Gröner Basis

- **System of polynomials**

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- **System of polynomials** - It can be shown that computing Gröbner basis using the lex ordering gives a basis where the variables are eliminated successively. Also, the order of elimination seems to correspond to the ordering of the variables.

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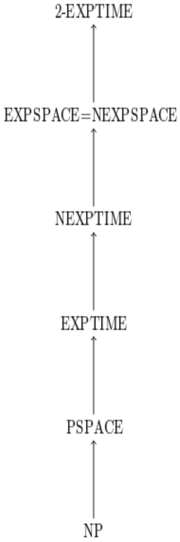
- **The Implicitization Problem** Similarly, we can eliminate the t variables and the rest of the equations define the ideal we require. Example,

$$I = (t^4 - x, t^3 - y, t^2 - z)$$
$$G = \{t^2 + z, ty - z^2, tz - y, x - z^2, y^2 - z^3\}$$

Thus, $(x - z^2, y^2 - z^3)$ is the required ideal.

Complexity



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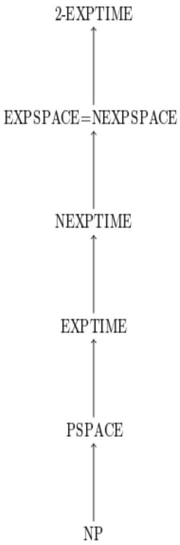

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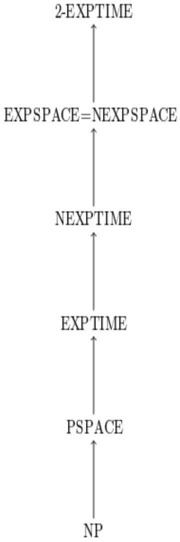

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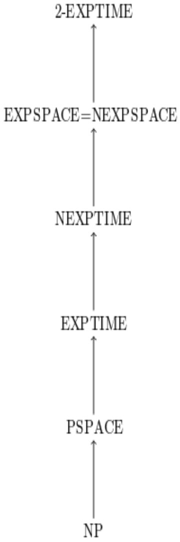
- Ideal membership problem is EXPSPACE-complete [Mayr-Meyer'82]
- Polynomial System solving is in PSPACE . [Koll´ar'88, Fitchas-Galligo'90]

Complexity



- The worst case time complexity of Buchberger's algorithm is $O(2^{2^n})$ time which restricts its usage.
- Ideal membership problem is EXPSPACE-complete [Mayr-Meyer'82]
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- However, better algorithms can be constructed for specific purposes. For example, computing a Gröbner basis for the radical of a zero dimensional Ideal takes randomized $O(d)$, deterministic $O(d^n)$ time. [Lakshman '90]

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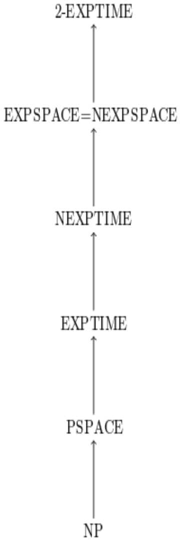

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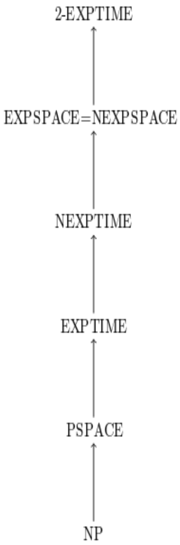

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- Faster Algorithms by Jean-Charles Faugére (F_4, F_5) for a certain (broad) class of systems called *regular sequences* in singly exponential time. Quite fast in the general case as well, used in computer algebra systems.

Applications

- Effective computation with (holonomic) special functions
- Solving Diophantine equations (Pell)
- Automated geometry theorem proving.
- Coding theory
- Signal and image processing
- Robotics
- Graph coloring problems e.g. Sudoku puzzles
- Extrapolating "missing links" in palaeontology, and phylogenetic tree construction

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