The statement

**Theorem 1.** Let \( f_i \in \mathbb{R}[x_1, \cdots, x_n] \ i \in [m] \). Then the number of connected components of the locus defined by \( f_i \geq 0 \) is bounded by \( \frac{1}{2} (D + 2)(D + 1)^{n-1} \) where \( D = \sum_i \deg f_i \)

An important lemma

**Lemma 2.** Number of non-degenerate solutions of \( f_1 = f_2 = \cdots f_m = 0 \) is at most \( \prod_i \deg f_i \)

**Proof.** Important thing is that since \( \mathbb{R} \) is not algebraically closed we can't use Bezout's theorem directly. Let \( a \) be a non-degenerate solution. By definition its Jacobian is non-singular and viewing \( f_i \) now as elements of \( \mathbb{C}[\bar{x}] \), the inverse function theorem says that \( a \) is an isolated root. Now, we may apply Bezout's theorem and conclude that the number of such \( a \)'s is \( \leq \prod_i \deg f_i \)

Milnor-Thom for a hypersurface

Before we do this there is an important technical proposition we need.

**Proposition 3.** If \( f \) is such that \( \nabla(f)(x) \neq 0 \ \forall x \in V(f) \), then the system of equations, 
\[ f = \partial_1 f = \cdots \partial_{n-1} f = 0 \]
has only non-degenerate solutions.

**Proof.** Direct from [Burgisser et al., 2010]. Let \( V := Z(f, \partial_1 f, \cdots, \partial_{n-1} f) \). Define \( g : V \to S^{n-1} \) as \( g(x) = \frac{\nabla f}{||\nabla f||}(x) \). Since the graph of \( g \) can be realized as a semi-algebraic set defined by 
\[ \{ y_i \partial_i f(x) \geq 0, \ y_i^2 ||\nabla f(x)||^2 = \partial_i f(x)^2 \ | \ i \in [n] \} \]
By the semi-algebraic Morse-Sard theorem we have that the space of the critical values has dimension \( < n \). Thus, \( \exists w ( \text{ say } ) = (0, \cdots, 0, 1) \) such that \( w, -w \) are not critical values. Now, \( g^{-1}(w) \cup g^{-1}(-w) = V \) as the gradient is non-zero but the first \( n-1 \) partial derivatives are. Let \( \alpha \in V \). We need that \( \alpha \) is non-degenerate. For any \( x \in \mathbb{R}^n \), denote by \( x' \) its projection to first \( n-1 \) coordinates. Since, \( n^{th} \) derivative is non-zero we can use implicit function theorem to obtain a \( C^\infty \) function \( h \) such that the map \( x' \to (x', h(x')) \) is a diffeomorphism to a neighbourhood around \( \alpha \). Now, computing partial derivatives we obtain,
\[
\partial_i f(x', h(x')) = -\partial_n f(x', h(x')) \partial_i h(x') \ i < n \\
\implies \partial_i h(\alpha') = 0 \ i < n \\
\partial_i g_j(\alpha) = -\partial_{ij}^2 h(\alpha') \ i, j < n \\
\implies \partial_{i,j}^2 f(\alpha) = -\partial_n f(\alpha) \partial_{i,j}^2 h(\alpha') \ i, j < n
\]

**Theorem 4.** Let \( f \in \mathbb{R}[x_1, \cdots, x_n] \) be such that \( n \geq 2, \deg f \geq 2, \nabla f(x) \neq 0 \ \forall x \in V(f) \) and \( V(f) \) is compact, then, \( b_0(V(f)) \leq \frac{1}{2}d(d-1)^{n-1} \) where \( d := \deg f \)
Proof. Let \( V(f) = \bigcup_i V_i \) where \( V_i \) are the connected components. Since closed subsets of compact sets are compact, \( V_i \) is compact. Since, \( V_i \) is a connected component, it is irreducible and is therefore a variety. From the definition of dimension of a variety (See Mumford [1976]), since every point is smooth, its dimension is \( n - 1 \). Define \( \pi_n := (x_1, \ldots, x_n) \rightarrow x_n \).

Let \( p_i \) be the minimum of \( \pi_n \) on \( V_i \) and let \( q_i \) be the maximum. At both points the gradient should be along the derivative of \( \pi_n = (0, \ldots, 0, 1) \). Thus, each such point, \( p_i, q_i \in V(f, \partial_1 f, \ldots, \partial_n f) \). Moreover, \( \forall i p_i \neq q_i \) because if not, then that implies that \( \forall v \in V_i, \; \pi_n(v) = \pi_n(p_i) =: a \) Now, that means that \( V_i \subset V(x_n - a) \). But we know from above that \( \dim V_i = n - 1 \). Thus, \( V_i = V(x_n - a) \) which contradicts compactness of \( V_i \).

Now, \( b_0(V(f)) = \frac{1}{2} |\{p_i, q_i\}| \leq \frac{1}{2} |V(f, \partial_1 f, \ldots, \partial_n f)| \). By the above proposition all the zeroes are non-degenerate and by Lemma 2 we get the required bound. \( \square \)

Extending to semi algebraic sets defined by many polynomials

Let \( S = \{ \bar{x} \mid f_1(\bar{x}) \geq 0, \ldots, f_m \geq 0 \} \). Dealing with this presents us with 2 issues that prevents us from using the previous machinery - One is that is not a zeroset, and the other is it’s not necessarily compact. We solve the second issue first.

Solving non-compactness - Since we have a metric namely the Euclidean one on \( \mathbb{R}^n \) we simply look at \( S \cap B_r \) i.e. those points in \( S \) with distance from origin at most \( r \). This can be realized by adding another polynomial constraint \( f_0^r = r^2 - (\sum_i x_i^2) \geq 0 \). The following lemma shows that obtaining a bound for this restriction suffices.

Lemma 5. Let \( K_i \subset \mathbb{R}^n \; \forall i \in \mathbb{N} \) such that \( K_i \subset K_{i+1} \), then \( b_0(\bigcup_{i \in \mathbb{N}} K_i) \leq \sup_{i \in \mathbb{N}} b_0(K_i) \)

Proof. Let \( C_1, \ldots, C_s \) be the connected components of \( \bigcup_{i \in \mathbb{N}} K_i \). This implies that \( \exists k_j C_j \cap K_i \neq \emptyset \; \forall t \geq k_j \). Choose, \( m = \max_j k_j \). Now for each \( K_m \) and beyond, the intersection with each \( C_i \) is non-trivial. Moreover, they can’t merge into the same connected component. Thus, \( \sup_{i \in \mathbb{N}} b_0(K_i) \geq b_0(K_m) \geq s = b_0(\bigcup_{i \in \mathbb{N}} K_i) \) \( \square \)

Applying this lemma with \( K_n = S \cap B_n \) will give us that we need to just upper bound the compact set \( S \cap B_n \) for an arbitrary (but fixed) \( n \).

Making it a zeroset - To do this we modify \( S \cap B_r \) by adding an \( \epsilon \) to each \( f_i \) and adding the polynomial constraint \( f_{n+1} = \prod_i (f_i + \epsilon) \geq \delta , \epsilon \geq \epsilon^{n+1} \geq \delta > 0 \). Thus to clarify \( S_{r, \epsilon, \delta} := \{ \bar{x} \mid f_0^r + \epsilon \geq 0, \ldots, f_n + \epsilon \geq 0, f_{n+1} \geq \delta \} \). Let’s look at the boundary of \( S_{r, \epsilon, \delta} := \partial S \). At the boundary at least one of the inequalities should be tight and the point be in \( S \). But if any except the last is 0, the last inequality can’t hold. Thus the boundary is defined by \( \partial S = V(f_{n+1}) \). This is clearly compact. We can make it non-singular by choosing \( \delta \) appropriately (This is by Sard’s theorem as we need to choose a \( \delta \) such that it isn’t a critical value of \( f_{n+1} \) and that is possible as this set isn’t dense). Applying Theorem 4 we get that

\[
b_0(\partial S) \leq \frac{1}{2} (D + 2)(D + 1)^{n-1} , \quad D = \sum_{i=1}^{m} \deg f_i
\]
Proof of Milnor-Thom bound

To make this exercise meaningful, we need the following result.

**Lemma 6.** $b_0(S) \leq b_0(\partial S)$

**Proof.** Since connected components are disjoint if they meet at boundary they do so in different components. Thus, we are done if we show that each connected component $C$ of $S$ satisfies $C \cup \partial S \neq \phi$. For a contradiction assume $C$ doesn’t. Then, for each $x \in C$, we have $B_{r_x} \subseteq S \setminus \partial S$. The union of these is an open cover of $C$. Since, $S$ is compact so is $S$ and thus we have a finite subcover. Let $R$ be the min radius $r_x$ of this set of this finite set of balls. The open set $\{x \mid \text{dist}(x, C) < R\} \subseteq S \setminus \partial S$ is $\supseteq C$ and is clearly connected. This is a contradiction as a connected component is the maximal such. □

So we have that, $b_0(S_{r,\epsilon,\delta}) \leq \frac{1}{2}(D+2)(D+1)^{n-1}$. $S_r = \bigcap_{i<1} S_{r,\epsilon,\epsilon_{i+1}}$ To wrap up things we just need the other counterpart of 5 and that we mention without proof.

**Lemma 7.** Let $K_i \subset \mathbb{R}^n \forall i \in \mathbb{N}$ such that $K_i \supset K_{i+1}$, then $b_0(\cap_{i \in \mathbb{N}} K_i) \leq \lim_{i \to \infty} \inf b_0(K_i)$

### Blackboxes

**Theorem 8** (Bezout’s Inequality). The number of isolated solutions of $f_1 = f_2 = \cdots = f_n = 0$ are atmost $\prod_i \deg f_i$ where $f \in k[\bar{\mathbb{X}}]$ such that $k$ is algebraically closed.

**Theorem 9** (Semi-algebraic Morse-Sard). Direct from Burgisser et al. [2010]. Let $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^m$ be semi-algebraic subsets and smooth submanifolds, and let $\phi : V \to W$ be a smooth, semi-algebraic map. Let

$$\Sigma := \{a \in V \mid \text{rk } d_\phi a < W\}$$

denote the set of critical points of $\phi$. Then $\dim \phi(\Sigma)$ is $< \dim W$

**Theorem 10** (Implicit function theorem). Let $f$ be a $C^k$ map $U \subset \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n$. Let $p = (x, y) \in U$ such that the derivative $Df(p)$ restricted to first $n$ coordinates is invertible. Then there is a neighborhood $V \times W$ of $p$ and a $C^k$ smooth map $h : W \to V$ such that $x = h(y)$ and $f(h(y), y) = 0$

The above exposition follows basically the approach of Milnor [1964] but uses the elementary (avoiding Cech cohomology) proofs of certain results as in Burgisser et al. [2010] to prove the required result.

### References

